

Frequency-Domain Homogenization of Maxwell Equations in Complex Media

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ABSTRACT

This paper is devoted to the homogenization of the Maxwell equations with periodically oscillating coefficients in the bianisotropic media which represents the most general linear media. In the first time, the limiting homogeneous constitutive law is rigorously justified in the frequency domain and is found from the solution of a local problem on the unit cell. The homogenization process is based on the two-scale convergence conception. In the second time, the implementation of the homogeneous constitutive law by using the finite element method and the introduction of the boundary conditions in the discrete problem are introduced. Finally, the numerical results associated of the perforated chiral media are presented.

KEYWORDS

Maxwell equations — Homogenization — Bianisotropic materials — Metamaterials — Effective parameters — Finite element method (FEM) — Two-scale convergence.

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1. Introduction

Mathematical homogenization theory started in the late sixties and has been extensively developed during the last two decades. It is now a well established discipline in mathematics. Homogenization is widely used in various areas such as in composite engineering (heat conduction, elastic deformation, porous media, acoustics, ...), material science, geophysics, fluid mechanics, elasticity etc. From the homogenization point of view, non-homogeneous composite materials are characterized by two separate scales, the microscopic scale, describing the heterogeneities, and the macroscopic scale, describing the general behavior of the composite. Thus, the locally heterogeneous material behaves like a homogeneous medium when the characteristic size of the inclusions is much smaller than the size of the whole sample. In this case, the behavior of the composite can be characterized by the so-called effective parameters which are obtained by either an averaging or limit process.

An important number of homogenization approaches have been developed for evaluating the effective or homogenized electromagnetic response and they are based on different methods and approximation schemes. The simplest approaches employ the principle of retrieval. Indeed, the effective parameters are obtained and retrieved from the scattering properties of the medium [1]-[3] by postulating the equivalence between a complex artificial material and a uniform slab of same thickness with unknown constitutive parameters.

Another homogenization approach supposes that the studied artificial and composite domain is periodic. The effective constitutive parameters are expressed from the macroscopic electromagnetic properties (fields, inductions, ...). These macroscopic properties are obtained by averaging the electromagnetic field in a metamaterial unit cell [4]-[7]. It should be noted that the electromagnetic properties and the effective parameters are obtained by using the numerical methods as Finite Element Method (FEM) and Moment-Method (MM).

In addition to numerical methods, the analytical methods or the classical formalisms which are available where the effective parameters are evaluated from the distribution of the underlying metamaterial inclusions. The most popular formalisms, we cite exploiting Lorentz [8], Clausius Mossotti [9], Maxwell-Garnett [10] approximations, or based on multipolar expansion [11] and source-driven approach [12].

The mathematical approaches are considered to be more rigorous and are generally based on a limit process to study homogenization of boundary value problems with periodic rapidly oscillating coefficients. In 1989, the two scale convergence method was introduced by Nguetseng [13]-[14]. In 1992, Allaire introduced the name “two-scale convergence” [15] by proposing an excellent proof of Nguetseng’s compactness theorem and studied properties of the two-scale convergence method. After that in 1994, Bourgeat, Mikelic, and Right [16] introduced the “stochastic two-scale convergence” to study random homogenization. In 1996, the extension of two-scale convergence method to periodic surfaces was presented by Neuss-Radu [17]. The two-scale convergence has been studied in many other papers [18]-[26]. More recently in 2002, the “unfolding method” was introduced by Damlamian, Griso and Cioranescu [27]-[28] in order to study the homogenization of periodic heterogeneous composites while Nguetseng [29] extended the two-scale convergence method to tackle deterministic homogenization beyond the periodic setting. In 2009, Wellander [30] presented two-scale Fourier transform (for periodic homogenization in Fourier spaces) which connects some existing techniques for periodic homogenization, namely: the twoscale convergence method, the periodic unfolding method and the Floquet-Bloch expansion approach to homogenization.

The mathematical approaches were applied to homogenize the Maxwell equations in frequency and time domain [31]-[33] and the numerical results were presented for some classes of the electromagnetic materials [34]-[40]. In this work, we revisit the homogenization theory of Maxwell equations in the bianisotropic media which represents the most general linear media. This study presents different aspects. Indeed, we present the theoretical foundation of the homogenization of Maxwell equations, the numerical analysis of the problem and finally the numerical results of the homogenized parameters.

The paper is organized in the following way. After having given the prerequisites for the work in the second section, we derive in the third section the limiting homogeneous constitutive law in the frequency domain. In the fourth and the fifth sections, we present respectively the finite element discretization and the numerical implementation of the homogenized problem. In the sixth section, we give some numerical results of the homogenized constitutive parameters associated to a chiral perforated material. The paper is concluded by a series of appendices containing some of the mathematical notions used in the paper.

2. Heterogeneous problem

We assume that the domain $\Omega \subset \mathbb{R}^3$ is modeled by a α -periodic material in the three Cartesian coordinate directions, i.e., it is the union of a collection of identical cubes with side length α (Y^α -cells). Moreover, $Y^\alpha = \alpha Y$, where $Y = [0, 1]^3$ is the unit cube. Ω is bounded with Lipschitz boundary $\partial\Omega$. Under the action of exterior source (J^E, J^H) , the electromagnetic field solutions depend on the size α , and, therefore, all fields are indexed by the periodicity α . The fields (E^α, H^α) satisfy the time-harmonic Maxwell equations in Ω (time convention e^{-pt} where $p = i\omega$ is Laplace variable):

$$\begin{aligned} \text{curl } E^\alpha(p, x) &= p B^\alpha(p, x) + J^E(x), \\ \text{curl } H^\alpha(p, x) &= -p D^\alpha(p, x) + J^H(x). \end{aligned} \quad (1)$$

with the ideal conductor boundary condition

$$n(x) \times E^\alpha(p, x) = 0 \quad \text{on} \quad \partial\Omega \quad (2)$$

where n is the normal unit vector.

The constitutive relations relate the electromagnetic fields (E^α, H^α) to the electromagnetic inductions (D^α, B^α) . In the general case of bianisotropic electromagnetic material, these relations are expressed in the following way:

$$\begin{aligned} D^\alpha(p, x) &= \bar{\epsilon}^\alpha(x) E^\alpha(p, x) + \bar{\xi}^\alpha(x) H^\alpha(p, x), \\ B^\alpha(p, x) &= \bar{\zeta}^\alpha(x) E^\alpha(p, x) + \bar{\mu}^\alpha(x) H^\alpha(p, x). \end{aligned} \quad (3)$$

where the four material dyadics are : relative permittivity $\bar{\bar{\epsilon}}^\alpha$ and relative permeability $\bar{\bar{\mu}}^\alpha$, and two cross-polarization (magnetolectric) dyadics $\bar{\bar{\xi}}^\alpha$ and $\bar{\bar{\zeta}}^\alpha$. These constitutive parameters are periodic with a small scale period α with $\alpha > 0$; more precisely we assume that

$$\bar{\bar{\epsilon}}^\alpha(x) = \bar{\bar{\epsilon}}(x/\alpha), \quad \bar{\bar{\mu}}^\alpha(x) = \bar{\bar{\mu}}(x/\alpha), \quad \bar{\bar{\xi}}^\alpha(x) = \bar{\bar{\xi}}(x/\alpha), \quad \bar{\bar{\zeta}}^\alpha(x) = \bar{\bar{\zeta}}(x/\alpha). \quad (4)$$

where $\bar{\bar{\epsilon}}^\alpha$, $\bar{\bar{\mu}}^\alpha$, $\bar{\bar{\xi}}^\alpha$ and $\bar{\bar{\zeta}}^\alpha$ are periodic matrix-valued functions on \mathbb{R}^3 of common period Y . If the medium does not have any preferred direction, it is called bi-isotropic, and all dyadics are multiples of the unit dyadic I_3 . Physical restrictions set some conditions to the material dyadics. If no dissipation is allowed, the medium is lossless. Applied to bianisotropic media, these conditions mean that $\bar{\bar{\epsilon}}^\alpha = \bar{\bar{\epsilon}}^{\alpha*}$, $\bar{\bar{\mu}}^\alpha = \bar{\bar{\mu}}^{\alpha*}$ and $\bar{\bar{\xi}}^\alpha = \bar{\bar{\zeta}}^{\alpha*}$, where the hermitian operator $*$ denotes a complex conjugate of the transpose. If the bianisotropic material is reciprocal, these conditions become : $\bar{\bar{\epsilon}}^\alpha = \bar{\bar{\epsilon}}^{\alpha T}$, $\bar{\bar{\mu}}^\alpha = \bar{\bar{\mu}}^{\alpha T}$ and $\bar{\bar{\xi}}^\alpha = -\bar{\bar{\zeta}}^{\alpha T}$, where T is the transpose operator [41].

The permittivity $\bar{\bar{\epsilon}}^\alpha$, the permeability $\bar{\bar{\mu}}^\alpha$, and the two cross-polarization (magnetolectric) dyadics $\bar{\bar{\xi}}^\alpha$ and $\bar{\bar{\zeta}}^\alpha$ are in $L^\infty(\Omega)$ and there exist strictly positive constants c_1, c_2, c_3 and c_4 such that the following inequalities are verified:

$$\begin{aligned} \sum_{i,j=1}^3 \bar{\bar{\epsilon}}_{i,j}^\alpha(y) z_i z_j &\geq c_1 |z|^2 & \forall z \in \mathbb{R}^3 \\ \sum_{i,j=1}^3 \bar{\bar{\mu}}_{i,j}^\alpha(y) z_i z_j &\geq c_2 |z|^2 & \forall z \in \mathbb{R}^3 \\ \sum_{i,j=1}^3 \bar{\bar{\xi}}_{i,j}^\alpha(y) z_i z_j &\geq c_3 |z|^2 & \forall z \in \mathbb{R}^3 \\ \sum_{i,j=1}^3 \bar{\bar{\zeta}}_{i,j}^\alpha(y) z_i z_j &\geq c_4 |z|^2 & \forall z \in \mathbb{R}^3 \end{aligned} \quad (5)$$

Using the constitutive relations (3), the time harmonic Maxwell equations and the boundary condition, the electromagnetic problem can be written as follow:

$$\begin{cases} \text{curl } H^\alpha(p, x) &= p (\bar{\bar{\epsilon}}^\alpha(x) E^\alpha(p, x) + \bar{\bar{\xi}}^\alpha(x) H^\alpha(p, x)) + J^E(x) \\ \text{curl } E^\alpha(p, x) &= -p (\bar{\bar{\zeta}}^\alpha(x) E^\alpha(p, x) + \bar{\bar{\mu}}^\alpha(x) H^\alpha(p, x)) + J^H(x), \text{ in } \Omega \\ n(x) \times E^\alpha(p, x) &= 0 \text{ on } \partial\Omega \end{cases} \quad (6)$$

Let us introduce some appropriate definitions

- $H(\text{curl}; \Omega) = \{v \in L^2(\Omega; \mathbb{R}^3) : \text{curl } v \in L^2(\Omega; \mathbb{R}^3)\}$
- $H_0(\text{curl}, \Omega) = \{v \in H(\text{curl}; \Omega) : n(x) \times v(x) = 0\}$
- $V(\Omega) = H_0(\text{curl}, \Omega) \times H(\text{curl}; \Omega)$.

$H(\text{curl}; \Omega)$ is equipped with the following norm

$$\|v\|_{H(\text{curl}; \Omega)}^2 = \|v\|_{L^2(\Omega)}^2 + \|\text{curl } v\|_{L^2(\Omega)}^2$$

Proposition 2.1. *Let $\bar{\bar{\epsilon}}^\alpha(x)$, $\bar{\bar{\mu}}^\alpha(x)$, $\bar{\bar{\xi}}^\alpha(x)$ and $\bar{\bar{\zeta}}^\alpha(x) \in L^\infty(\Omega; \mathbb{R}^9)$. There exists $p_0 > 0$ such that the problem (6) has a unique solution $(E^\alpha, H^\alpha) \in L^\infty(p_0, \infty; V(\Omega))$. This solution satisfies the uniform bounds:*

$$\|E^\alpha(p)\|_{L^2(\Omega)} \leq \frac{c}{p} \quad \|\text{curl } E^\alpha(p)\|_{L^2(\Omega)} \leq c \quad \forall p > p_0 \quad (7)$$

$$\|H^\alpha(p)\|_{L^2(\Omega)} \leq \frac{c'}{p} \quad \|\text{curl } H^\alpha(p)\|_{L^2(\Omega)} \leq c' \quad \forall p > p_0 \quad (8)$$

The proof of this proposition is established in [33].

Subsequently, we work at given frequency p with $p > p_0$.

3. Homogenized problem

3.1 Effective parameters

The aim of this section is to study the behavior of the electromagnetic field solution of the problem (6) and the constitutive laws when the period of the microstructure goes to zero. The limiting behavior of the electromagnetic properties is obtained using the two-scale convergence method and stated in the next theorem.

Theorem 3.1. *The sequence of solutions $(E^\alpha, H^\alpha) \in V(\Omega)$ of (6) converges weakly into $(E, H) \in V(\Omega)$ the unique solution of the homogenized equations*

$$\begin{cases} \operatorname{curl} E(x) = p (\bar{\xi}^H(x)E(x) + \bar{\mu}^H(x)H(x) + J^E(x), \\ \operatorname{curl} H(x) = -p (\bar{\varepsilon}^H(x)E(x) + \bar{\xi}^H(x)H(x) + J^H(x), \\ n(x) \times E(x) = 0 \quad \text{on} \quad \partial\Omega \end{cases} \quad (9)$$

The homogenized constitutive parameters $\bar{\mu}^H$, $\bar{\varepsilon}^H$, $\bar{\xi}^H$ and $\bar{\xi}^H$ are described by their columns

$$\begin{aligned} \bar{\varepsilon}_k^H &= \int_Y [\bar{\varepsilon}(y)(e_k + \nabla_y \chi_k^1(y)) + \bar{\xi}(y) \nabla_y \chi_k^2(y)] dy \\ \bar{\mu}_k^H &= \int_Y [\bar{\mu}(y)(e_k + \nabla_y \chi_k^2(y)) + \bar{\xi}(y) \nabla_y \chi_k^1(y)] dy \\ \bar{\xi}_k^H &= \int_Y [\bar{\xi}(y)(e_k + \nabla_y \chi_k^2(y)) + \bar{\varepsilon}(y) \nabla_y \chi_k^1(y)] dy \\ \bar{\xi}_k^H &= \int_Y [\bar{\xi}(y)(e_k + \nabla_y \chi_k^1(y)) + \bar{\mu}(y) \nabla_y \chi_k^2(y)] dy \end{aligned} \quad (10)$$

with $\chi^1(y) = \chi_k^1(y)e_k$ and $\chi^2(y) = \chi_k^2(y)e_k$ where e_k is the k -th canonical basis of \mathbb{R}^3 and if we note e'_k , the k -th canonical basis of \mathbb{R}^6 and $\chi(y) = (\chi_1(y), \chi_2(y))$, the term $\chi_k(y)$, $i = 1, 2, \dots, 6$, in $H_{per}^1(Y)/\mathbb{C}$ solve the local elliptic problem

$$\int_Y \begin{pmatrix} \bar{\varepsilon}(y) & \bar{\xi}(y) \\ \bar{\xi}(y) & \bar{\mu}(y) \end{pmatrix} (e'_k + \nabla_y \chi_k(y)) \cdot \nabla_y w(y) dy = 0 \quad \forall w \in H_{per}^1(Y; \mathbb{R}^2) \quad (11)$$

Proof.

The limiting homogeneous constitutive law is justified at given frequency by using the concept of two-scale convergence [13], see definition in appendix.

We introduce the following notations

- A^α is the associated matrix to the constitutive parameters of the material Ω

$$A^\alpha = \begin{pmatrix} \bar{\varepsilon}^\alpha & \bar{\xi}^\alpha \\ \bar{\xi}^\alpha & \bar{\mu}^\alpha \end{pmatrix}$$

- Let us denote \mathcal{E}^α the electromagnetic six-vector field, \mathcal{D}^α the electromagnetic displacement six-vector, and \mathcal{J} the six-vector source exterior given, respectively, by

$$\mathcal{E}^\alpha = \begin{pmatrix} E^\alpha \\ H^\alpha \end{pmatrix}, \quad \mathcal{D}^\alpha = \begin{pmatrix} D^\alpha \\ B^\alpha \end{pmatrix}, \quad \mathcal{J} = \begin{pmatrix} J^E \\ J^H \end{pmatrix}$$

- M represents the Maxwell operator from $L^2(\Omega; \mathbb{R}^6)$ to \mathbb{R}^6 and it can be written as follows

$$M : \mathcal{V} = (v_1, v_2) \in L^2(\Omega; \mathbb{R}^6) \rightarrow Mv = \begin{pmatrix} \operatorname{curl} v_2 \\ -\operatorname{curl} v_1 \end{pmatrix}$$

- N is an operator defined from $H_{per}^1(Y, \mathbb{R}^2)$ to \mathbb{R}^6 and expressed as:

$$N : w = (w_1, w_2) \in H_{per}^1(Y, \mathbb{R}^2) \rightarrow Nw = \begin{pmatrix} \nabla_y w_2 \\ -\nabla_y w_1 \end{pmatrix}$$

Hence the problem (6) can be written as:

$$\begin{cases} M\mathcal{E}^\alpha(x) = pA^\alpha(x)\mathcal{E}^\alpha(x) + \mathcal{J}(x) & \text{in } \Omega, \\ n(x) \times E^\alpha(x) = 0, & \text{on } \partial\Omega \end{cases} \quad (12)$$

Let $\phi(x) = \alpha w(x/\alpha)\mathcal{V}(x)$, where $w = (w_1, w_2) \in H_{per}^1(Y, \mathbb{R}^2)$ and $\mathcal{V} = (v_1, v_2) \in C_0^\infty(\Omega, \mathbb{C}^6)$. Then $\phi \in V(\Omega)$ is admissible test functions. We get in (12) and we obtain the following integral equation

$$\begin{aligned} \int_{\Omega} \mathcal{E}^\alpha(x) \cdot \{ \alpha w(x/\alpha)M\mathcal{V}(x) + Nw(x/\alpha) \times \mathcal{V}(x) \} dx = \\ p \int_{\Omega} \alpha w(x/\alpha)\mathcal{V}(x) \cdot (A(x/\alpha)\mathcal{E}^\alpha(x) + \mathcal{J}(x)) dx \end{aligned} \quad (13)$$

In the limit $\alpha \rightarrow 0$ we get

$$\int_{\Omega} \mathcal{E}^\alpha(x) \cdot Nw(x/\alpha) \times \mathcal{V}(x) dx \rightarrow 0 \quad (14)$$

The electromagnetic field \mathcal{E}^α is uniformly bounded in $L^2(\Omega)^2$ for fixed p (see Proposition 2.1), then we have two types of convergence, the first is in the sense of the weak convergence and the second is in the two-scale convergence.

1. The boundedness of A^α together with (7) for fixed p imply that \mathcal{D}^α is also bounded in $(L^2(\Omega))^2$, then two subsequences (still denoted by α) can be extracted from α such that, letting $\alpha \rightarrow 0$, there holds

$$\mathcal{E}^\alpha \rightharpoonup \mathcal{E} \quad \text{weakly in } (L^2(\Omega))^2 \quad (15)$$

$$\mathcal{D}^\alpha \rightharpoonup \mathcal{D} \quad \text{weakly in } (L^2(\Omega))^2 \quad (16)$$

We combine (1), (15) and (16), the vector field $\nabla \times \mathcal{E}^\alpha$ has L^2 norm that is still bounded as $\alpha \rightarrow 0$. So, it has weak limit in $(L^2(\Omega))^2$. Hence the limit (16) implies that \mathcal{E} belongs to $H(\text{curl}, \Omega)^2$ and

$$\mathcal{E}^\alpha \rightharpoonup \mathcal{E} \quad \text{weakly in } (H(\text{curl}, \Omega))^2 \quad (17)$$

2. There exists a subsequence of \mathcal{E}^α which converges in the two-scale sense. We will keep the index α for this subsequence. The use of the theorem 8.1 (see appendix) and (14), allow us to write

$$\int_{\Omega} \int_Y \mathcal{E}_0(x, y) \cdot Nw(y) \times \mathcal{V}(x) dy dx = 0 \quad (18)$$

After a cyclic permutation, the equation obtained is

$$\int_Y \mathcal{E}_0(x, y) \times Nw(x) dy = 0 \quad (19)$$

for all $w \in H_{per}^1(Y; \mathbb{R}^2)$.

The function $\mathcal{E}_0(x, y)$ belongs to the space $L^2(\Omega; L_{per}^2(Y; \mathbb{C}^6))$. From lemma 8.2 (see appendix), we conclude that the field $\mathcal{E}_0(x, y)$ can be decomposed as

$$\mathcal{E}_0(x, y) = \mathcal{E}(x) + \nabla_y \psi(x, y) \quad (20)$$

where $\mathcal{E}(x)$ is the average of the $\mathcal{E}_0(x, y)$ upon the unit cell Y .

In summary

$$\mathcal{E}^\alpha(x) \xrightarrow{2-s} \mathcal{E}(x) + \nabla_y \psi(x, y) \quad (21)$$

Multiplication of (12) by the admissible test function $\phi \in C_0^\infty(\Omega; \mathbb{C}^6)$ gives

$$\int_{\Omega} M\mathcal{E}^\alpha(x) \cdot \phi(x) dx = p \int_{\Omega} (A(x/\alpha)\mathcal{E}^\alpha(x) + \mathcal{J}(x)) \cdot \phi(x) dx \quad (22)$$

By using theorem 8.4 (see appendix), the two-scale convergence of $M\mathcal{E}^\alpha(x)$ is

$$M\mathcal{E}^\alpha(x) \xrightarrow{2-s} M_x \mathcal{E}_0(x, y) + M_y \mathcal{E}_1(x, y) = M\mathcal{E}(x) + M_y \mathcal{E}_1(x, y) \quad (23)$$

Since the admissible test function ϕ does not depend on y , at the limit $\alpha \rightarrow 0$, we get

$$\int_{\Omega} M^{\mathcal{E}}(x) \cdot \phi(x) dx = p \int_{\Omega} \int_Y (A(y)(\mathcal{E}(x) + \nabla_y \psi(x,y)) + \mathcal{J}(x)) \cdot \phi(x) dy dx \quad (24)$$

The divergence equation $(A(x/\alpha)\mathcal{E}^{\alpha}(x) + \mathcal{J}(x))$ is equal to zero

$$\nabla \cdot (A(x/\alpha)\mathcal{E}^{\alpha}(x) + \mathcal{J}(x)) = \nabla \cdot M^{\mathcal{E}^{\alpha}}(x) = 0 \quad (25)$$

We multiply $(\nabla \cdot (A(x/\alpha)\mathcal{E}^{\alpha}(x) + \mathcal{J}(x)))$ by a test function $\mathcal{V}(x) = \alpha \psi(x) \phi(x/\alpha)$ where $\psi \in C_0^{\infty}(\Omega, \mathbb{R}^6)$ and $\phi \in H_{per}^1(Y; \mathbb{R}^2)$. If we note that $w_A(y) = e_i A(y) e_j \in L_{per}^{\infty}(Y)$ then $w_A(y) \nabla_y \phi$ is in $L_{per}^2(Y; \mathbb{C}^6)$. By using theorem 8.1 (see appendix), we obtain the following relation:

$$\begin{aligned} & \lim_{\alpha \rightarrow 0} \int_{\Omega} \nabla(A(x/\alpha)\mathcal{E}^{\alpha} + \mathcal{J}(x)) \cdot \alpha \psi(x) \phi(x/\alpha) dx \\ &= - \lim_{\alpha \rightarrow 0} \int_{\Omega} (A(x/\alpha)\mathcal{E}^{\alpha}(x) + \mathcal{J}(x)) \cdot (\alpha \nabla \psi(x) \phi(x/\alpha) + \psi(x) \cdot \nabla_y \phi(x/\alpha)) dx \\ &= - \int_{\Omega} \int_Y \psi(x) \nabla_y \phi(y) \cdot A(y)(\mathcal{E}(x) + \nabla_y \psi(x,y)) dy dx = 0 \end{aligned} \quad (26)$$

for all $\psi \in H_0^1(\Omega)$. Then the local equation can be obtained and expressed as

$$\int_Y \nabla_y \phi(y) \cdot A(y)(\mathcal{E}(x) + \nabla_y \psi(x,y)) dy = 0 \quad (27)$$

The microscopic variable and the macroscopic variable can be separated by using the Ansatz

$$\nabla_y \psi(x,y) = \nabla_y \chi(y) \cdot \mathcal{E}(x) = \nabla_y \chi_k(y) \mathcal{E}_k(x) \quad (28)$$

where $\chi(y) = \chi_k(y) e'_k$ and $\chi_k(y) \in H_{per}^1(Y; \mathbb{R}^2)$

By using the Ansatz(28) in the local equation (27), this last becomes:

$$\int_Y (A(y)(e'_k + \nabla_y \chi_k(y))) \cdot \nabla_y \phi(y) dy = 0 \quad (29)$$

for all $\phi \in H_{per}^1(Y)$, then

$$\nabla_y (A(y)(e'_k + \nabla_y \chi_k(y))) = 0 \quad (30)$$

It can be checked (see lemma 8.3 in appendix) that there exists a unique solution of this equation up to a constant. Inserting the solution of the equation (28) into (24) yields the macroscopic homogenized equation

$$\int_{\Omega} M^{\mathcal{E}}(x) \cdot \phi(x) dx = p \int_{\Omega} \left(\int_Y A(y)(e'_k + \nabla_y \chi_k(y)) dy \right) \mathcal{E}_k(x) + \mathcal{J}(x) \phi(x) dx$$

which defines the effective constitutive matrix A^H as

$$A_k^H = \int_Y A(y)(e'_k + \nabla_y \chi_k(y)) dy, \quad \text{for } k = 1, \dots, 6 \quad (31)$$

where A^H contains the homogenized constitutive parameters $\bar{\mu}^H$, $\bar{\varepsilon}^H$, $\bar{\zeta}^H$ and $\bar{\xi}^H$

$$\begin{aligned} \bar{\varepsilon}_k^H &= \int_Y (\bar{\varepsilon}(y)(e_k + \nabla_y \chi_k^1(y)) + \bar{\xi}(y) \nabla_y \chi_k^2(y)) dy \\ \bar{\mu}_k^H &= \int_Y (\bar{\mu}(y)(e_k + \nabla_y \chi_k^2(y)) + \bar{\zeta}(y) \nabla_y \chi_k^1(y)) dy \\ \bar{\zeta}_k^H &= \int_Y (\bar{\zeta}(y)(e_k + \nabla_y \chi_k^2(y)) + \bar{\varepsilon}(y) \nabla_y \chi_k^1(y)) dy \\ \bar{\xi}_k^H &= \int_Y (\bar{\xi}(y)(e_k + \nabla_y \chi_k^1(y)) + \bar{\mu}(y) \nabla_y \chi_k^2(y)) dy \end{aligned} \quad (32)$$

The homogenized constitutive relations can be expressed as a function of the macroscopic electromagnetic properties. Indeed, the macroscopic inductions are expressed by using the macroscopic fields and the homogenized constitutive parameters as

$$D = \bar{\bar{\epsilon}}^H E + \bar{\bar{\xi}}^H H \quad (33)$$

$$B = \bar{\bar{\zeta}}^H E + \bar{\bar{\mu}}^H H \quad (34)$$

The equation (31) can be written as

$$M\mathcal{E}(x) = p \mathcal{E}(x) + \mathcal{J}(x) \quad (35)$$

Then, it remains to establish that the boundary condition $n \times E = 0$ is also satisfied. We consider a fix function $\varphi \in H^{\frac{1}{2}}(\partial\Omega)$. There exists $\phi \in H^1(\Omega)$ such that $\phi|_{\partial\Omega} = \varphi$ [42]. Now, for $\alpha > 0$ we have

$$\int_{\Omega} \text{curl } \phi \cdot E^\alpha = \int_{\Omega} \text{curl } E^\alpha \cdot \phi + \int_{\partial\Omega} \phi \cdot (n \times E^\alpha) \quad (36)$$

$$\int_{\Omega} \text{curl } \phi \cdot E = \int_{\Omega} \text{curl } E \cdot \phi + \int_{\partial\Omega} \phi \cdot (n \times E) \quad (37)$$

and we have the relations

$$\int_{\Omega} \text{curl } \phi \cdot E^\alpha \rightarrow \int_{\Omega} \text{curl } \phi \cdot E \quad (38)$$

$$\int_{\Omega} \text{curl } E^\alpha \cdot \phi \rightarrow \int_{\Omega} \text{curl } E \cdot \phi \quad (39)$$

Knowing that $n \times E^\alpha|_{\partial\Omega} = 0$ and from the equations (36) and (39), we have

$$\int_{\partial\Omega} \phi \cdot (n \times E) = \int_{\partial\Omega} \phi \cdot (n \times E) = 0 \quad \forall \phi \in H^{\frac{1}{2}}(\partial\Omega) \quad (40)$$

The final result is $n \times E|_{\partial\Omega} = 0$.

We see that, in general, the homogenized medium characterized by $(\bar{\bar{\epsilon}}^H(y), \bar{\bar{\mu}}^H(y), \bar{\bar{\xi}}^H(y), \bar{\bar{\zeta}}^H(y))$ is bianisotropic even though the medium $(\bar{\bar{\epsilon}}(y), \bar{\bar{\mu}}(y), \bar{\bar{\xi}}(y), \bar{\bar{\zeta}}(y))$ of the unit cell is biisotropic, i.e., the constitutive parameters are proportional to the identity dyadic

$$\begin{aligned} \bar{\bar{\epsilon}}(y) &= \varepsilon(y)I_3, & \bar{\bar{\mu}}(y) &= \mu(y)I_3 \\ \bar{\bar{\xi}}(y) &= \xi(y)I_3, & \bar{\bar{\zeta}}(y) &= \zeta(y)I_3 \end{aligned}$$

where $\varepsilon(y)$, $\mu(y)$, $\xi(y)$ and $\zeta(y)$ are scalars. □

3.2 Properties of the homogenized parameters

The mean consequence of theorem 3.1 is that the homogenized parameters are independent of electromagnetic properties of the domain Ω and of the incident field. Furthermore, the homogenized material properties satisfy some new assumptions as the heterogeneous parameters do, i.e., they are bounded ($\bar{\bar{\epsilon}}^H, \bar{\bar{\mu}}^H, \bar{\bar{\xi}}^H, \bar{\bar{\zeta}}^H \in L^\infty(\Omega; \mathbb{R}^9)$). If the medium is lossless, i.e, the heterogeneous material parameters tensor is hermitian ($A^* = A$), then the homogenized material is also lossless as proved below.

Proposition 3.2. *If the heterogeneous material is lossless then the homogeneous material is also lossless.*

Proof.

For all $\omega \in H_{per}^1(Y)$, the local problem (30) can be written as, ($k=1, \dots, 6$)

$$\int_Y \nabla_y \omega(y) \cdot A(y) e_k dy = - \int_Y \nabla_y \omega(y) \cdot A(y) \nabla_y \chi^k(y) dy,$$

We rewrite this equation in one set of equations by using $\chi = (\chi^1, \chi^2, \dots, \chi^5, \chi^6)$, we have

$$\int_Y \nabla_y \omega(y) \cdot A(y) dy = - \int_Y \nabla_y \omega(y) \cdot A(y) \nabla_y \chi(y) dy, \quad \forall \omega \in H_{per}^1(Y).$$

Since A is hermitian and by taking the test function $\omega = \chi^k$, we obtain

$$\int_Y A(y) \nabla_y \chi(y) dy = - \int_Y (\nabla_y \chi(y))^* \cdot A(y) \nabla_y \chi(y) dy,$$

The matrix A^H of the homogenized constitutive parameters, given in (31), can be written as

$$\begin{aligned} A^H &= \int_Y A(y) dy + \int_Y A(y) \nabla_y \chi(y) dy, \\ &= \int_Y A(y) dy - \int_Y (\nabla_y \chi(y))^* \cdot A(y) \nabla_y \chi(y) dy, \end{aligned}$$

The heterogeneous material is lossless ($A = A^*$),

$$\begin{aligned} A^H &= \int_Y A^*(y) dy - \int_Y (\nabla_y \omega(y))^* \cdot A^*(y) \nabla_y \chi(y) dy, \\ &= (A^H)^* \end{aligned}$$

So, the homogenized material is hermitian. Then, if the heterogeneous material is lossless then the homogeneous material is also lossless. □

4. Finite element discretization

In the previous section, the continuous problem has been proved to be well-posed (30) and can be discretized in space by the finite element method. This method is the most popular technique to solve the elliptic problems.

Let \mathcal{T}_h be a family of tetrahedrization of the unit cell Y into a finite number of elements K such that:

$$\overline{\bigcup_{K \in \mathcal{T}_h} K} = \bar{Y}$$

with $h = \max\{\text{diam}(K) | K \in \mathcal{T}_h\}$ is regular according to Ciarlet [43].

We denote by $\mathcal{T}_h^{\partial Y}$ the discretization of ∂Y that are obtained by taking the trace of \mathcal{T}_h on ∂Y . To express the periodic boundary conditions of the sub-corrector χ on the unit cell ∂Y , the discretization $\mathcal{T}_h^{\partial Y}$ is created in such a way on the opposite sides (see Fig. 1). Indeed, we note the nodes of the $\mathcal{T}_h^{\partial Y}$ by \mathcal{N}_L and we can write this last as: $\mathcal{N}_L = \mathcal{N}_{(x=0)} \cup \mathcal{N}_{(x=1)} \cup \mathcal{N}_{(y=0)} \cup \mathcal{N}_{(y=1)} \cup \mathcal{N}_{(z=0)} \cup \mathcal{N}_{(z=1)}$ where $\mathcal{N}_{(x=0)}$, $\mathcal{N}_{(x=1)}$, $\mathcal{N}_{(y=0)}$, $\mathcal{N}_{(y=1)}$, $\mathcal{N}_{(z=0)}$ and $\mathcal{N}_{(z=1)}$ are the set of nodes located on the facets $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$ and $z = 1$ respectively.

$$\begin{aligned} \forall \gamma \in \mathcal{A}_{x=0} \quad \exists \delta \in \mathcal{A}_{x=1} \quad / \quad \delta = \tau_1 \gamma \quad \text{with} \quad \tau_1 = (1, 0, 0) \\ \forall \zeta \in \mathcal{A}_{y=0} \quad \exists \eta \in \mathcal{A}_{y=1} \quad / \quad \zeta = \tau_2 \eta \quad \text{with} \quad \tau_2 = (0, 1, 0) \\ \forall \rho \in \mathcal{A}_{z=0} \quad \exists \sigma \in \mathcal{A}_{z=1} \quad / \quad \rho = \tau_3 \sigma \quad \text{with} \quad \tau_3 = (0, 0, 1) \\ \text{card}(\mathcal{A}_{x=0}) = \text{card}(\mathcal{A}_{x=1}), \quad \text{card}(\mathcal{A}_{y=0}) = \text{card}(\mathcal{A}_{y=1}) \quad \text{and} \quad \text{card}(\mathcal{A}_{z=0}) = \text{card}(\mathcal{A}_{z=1}) \end{aligned}$$

The degrees of freedom are the values of χ at the vertices of a mesh based on tetrahedra. We denote by a_k the node of tetrahedron K , and we define the sets of global nodes Ξ

$$\Xi = \{a_k | K \in \mathcal{T}_h\}$$

and its subsets

$$\Xi^0 = \{a_k | K \in \mathcal{T}_h, a_k \notin \partial Y\}$$

$$\Xi^{\partial Y} = \{a_k | K \in \mathcal{T}_h, a_k \in \partial Y\}$$

Let \hat{K} be the reference tetrahedron, such that each element $K \in \mathcal{T}_h$ is the image of \hat{K} under an inversible affine mapping $F_K : \hat{K} \rightarrow K$ of the form

$$F_K(\hat{x}) = T_K \hat{x} + b_K$$

The mapping F_K is a bijection between \hat{K} and K , T_K is an inversible 3×3 matrix and b_K is a translation vector.

Over each element K , we define a finite-dimensional function space X_h by

$$X_h = \{v_h \in C^0(\bar{\Omega}) \mid v_h \circ F_K \in \mathcal{P}_k(\hat{K}) \quad \forall K \in \mathcal{T}_h\} \cap H_{per}^1(\Omega)$$

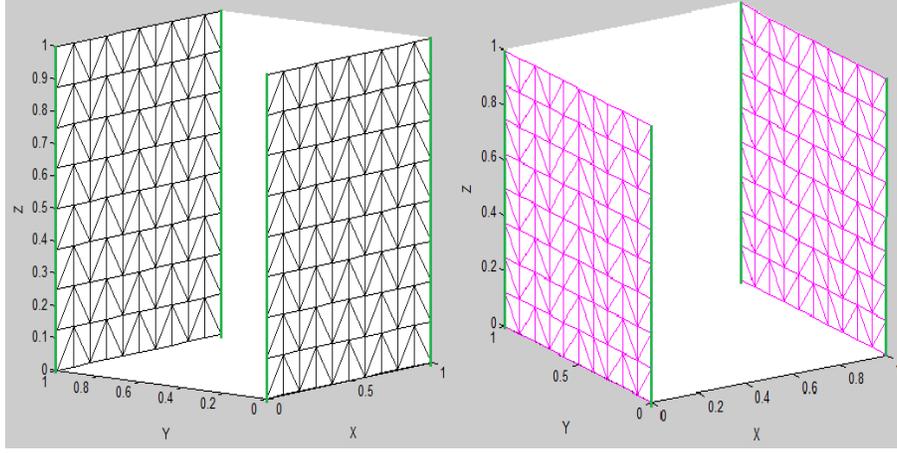


Figure 1. Opposite sides of the unit cell are meshed in the same way.

Let denote by φ_{a_j} the basis function associated to the j^{th} node of Ξ . These functions are periodic over $\Xi^{\partial Y}$

$$\begin{cases} \varphi_{a_j}(a_s) = \delta_{js} & \forall a_s \in \Xi^0, \\ \varphi_{a_j}(a_s) = \delta_{js}, \quad \varphi_{a_j}(a_s) = \varphi_{a_{\tau_j}}(a_s) & \forall a_s \in \Xi^{\partial Y} \end{cases}$$

where δ_{ij} is the Kronecker symbol ($\delta_{js} = 0$ for $j \neq s$ and $\delta_{jj} = 1$), and $a_{\tau_j} \in \Xi^{\partial Y}$ is the homologue node of the j^{th} node on the opposite surface of the ∂Y . To simply express the periodic boundary conditions on the sidewalls of the unit cell, the tetrahedral elements of the domain are created in such a way that the meshes on opposite surfaces of are identical.

We define the following approximation space:

$$\mathcal{U}_h = \{v_h \in X_h \mid v_h(a_j) = v_h(a_{\tau_j}) \quad \forall a_j \in \Xi^{\partial Y}\}$$

We propose to solve the following discrete problem associated to continuous problem (29):

$$\begin{aligned} \text{Find} \quad & \chi_h^k \in \mathcal{U}_h \times \mathcal{U}_h \\ & a(\chi_h^k, v) = f^k(v) \quad \forall v \in X_h \times X_h, \\ & \text{for } k = 1, \dots, 6. \end{aligned} \tag{41}$$

where

$$a(\chi_h^k, v) = \int_Y A(y) \cdot \nabla_y \chi_h^k(y) \cdot \nabla_y v(y) dy$$

and

$$f^k(v) = - \int_Y A(y) \cdot e'_k \cdot \nabla_y v(y) dy$$

Proposition 4.1. *There exists a unique χ_h^k solution of problem (41) up to a constant.*

Due to conformal approximation and to the uniqueness of the solution of the continuous problem (30), the approximate problem (41) also has a unique solution.

5. Implementation aspects

Concerning the spatial discretization, we have presented the finite element method for periodic geometries. Now, we focus our attention, firstly, on the numerical treatment of the sub-corrector χ_h^k solution of the equation (41) for $k = 1, \dots, 6$, involved in the definition of a basis for space $\mathcal{U}_h \times \mathcal{U}_h$. Secondly, we present the evaluation of the discrete approximation of the homogenized constitutive parameters.

5.1 Matrix form of the discretized problem

In this section, we aim at writing problem (41) in a matrix form. We construct a basis of an approximation space \mathcal{U}_h . It involves a linear system ($Qx^k = f^k$) whose solution represents the degrees of Freedom of χ_h^k .

By the resolution of final matrix system, we get the solution x^k for $k = 1, \dots, 6$ and the numerical solution of the discrete problem can be built directly by expanding the χ_h^k for $k = 1, \dots, 6$ in terms of the basis functions of the approximation space $X_h \times X_h$

$$\chi_h^k = \sum_{\{i/a_i \in \Xi\}} x_i^k \varphi_{a_i}(y). \quad (42)$$

We see that the discrete problem (41) is equivalent to the following linear system for the unknowns x_1^k, \dots, x_N^k where N is the number of the nodes in the mesh ($N = \text{card}(\Xi)$)

$$Qx^k = f^k, \quad (43)$$

where

$$\begin{aligned} x^k &= (x_1^k, \dots, x_N^k)^T \\ f^k &= - \left(\int_Y A(y) e_k \cdot \nabla_y \varphi_{a_1}(y) dy, \dots, \int_Y A(y) e_k \cdot \nabla_y \varphi_{a_N}(y) dy \right)^T \\ Q &= \left(\int_Y A(y) \nabla_y \varphi_{a_i} \cdot \nabla_y \varphi_{a_j} dy \right)_{1 \leq i, j \leq N} \end{aligned}$$

Let $a_i \in \Xi^{\partial Y}$ and $a_{\tau i} \in \Xi^{\partial Y}$ its homologue on the opposite surface. The degrees of freedom x_i^k and $x_{\tau i}^k$ associated to a_i and $a_{\tau i}$ verify the following expression

$$x_i^k = x_{\tau i}^k \quad (44)$$

Moreover, due to the basis properties the components of the second member f_i^k and $f_{\tau i}^k$ associated, respectively, to nodes a_i and $a_{\tau i}$ verify the following equation

$$\begin{aligned} f_{\tau i}^k &= - \int_Y A(y) e_k \cdot \nabla_y \varphi_{\tau i}(y) \\ &= - \int_Y A(y) e_k \cdot \nabla_y \varphi_i(y) dy \\ &= f_i^k \end{aligned} \quad (45)$$

Now we introduce the periodicity conditions related to the degree of freedom x^k (44) and of the second member f^k (45) of the two associated nodes a_i and $a_{\tau i}$ in the matrix system (43). We first start by introducing the condition ($x_i^k = x_{\tau i}^k$), i.e, we replace $x_{\tau i}^k$ by x_i^k in the vector x^k

$$Q (x_1^k, \dots, x_i^k, \dots, x_{\tau i-1}^k, x_i^k, x_{\tau i+1}^k, \dots, x_n^k)^T = f^k \quad (46)$$

The component x_i^k appears twice in the vector x^k . So, in matrix Q , we replace the column C_i by the column $C_i + C_{\tau i}$ by eliminating the column $C_{\tau i}$ and we eliminate the τi^{th} component of the vector x^k . The system (46) is equivalent to the following system

$$\left(C_1^Q, \dots, C_i^Q + C_{\tau i}^Q, \dots, C_{\tau i-1}^Q, C_{\tau i+1}^Q, \dots, C_N^Q \right) \cdot (x_1^k, \dots, x_i^k, \dots, x_{\tau i-1}^k, x_{\tau i+1}^k, \dots, x_n^k)^T = f^k \quad (47)$$

Finally, we insert the equations associated to the second member ($f_i^k = f_{\tau i}^k$) in the last system. The result is follows

$$\tilde{Q} x^k = (f_1^k, \dots, f_i^k, \dots, f_{\tau i-1}^k, f_i^k, f_{\tau i+1}^k, \dots, f_n^k)^T \quad (48)$$

We notice that the component f_i^k appears twice in the vector f^k . We substitute the line L_i by $L_i - L_{\tau i}$ in the matrix \tilde{Q} and we eliminate the line $L_{\tau i}$. In the second member f^k , we replace the i^{th} component by zero and we eliminate the τi^{th} one. The equivalent reduced system is

$$\left(L_1^{\tilde{Q}}, \dots, L_i^{\tilde{Q}} - L_{\tau i}^{\tilde{Q}}, \dots, L_{\tau i-1}^{\tilde{Q}}, L_{\tau i+1}^{\tilde{Q}}, \dots, L_N^{\tilde{Q}} \right) x^k = (f_1^k, \dots, 0, \dots, f_{\tau i-1}^k, f_{\tau i+1}^k, \dots, f_n^k)^T \quad (49)$$

By introducing all periodicity relations linking the whole degrees of freedom and the components of second member, we obtain the following linear system characterized by reduced size compared to the size of initial system

$$\bar{Q}\bar{x}^k = \bar{f}^k, \quad (50)$$

where \bar{Q} is $2l \times 2l$ matrix, \bar{f}^k is $2l$ vector and l is the number of non interlinked nodes. The matrix Q in the system (43) is symmetric and sparse. In spite of the introduction of periodic conditions in Q , the obtained matrix, \bar{Q} , is still sparse but loses symmetry aspect. The resolution of the system (50) can be done either by a direct method such as Cholesky factorization or by an iterative one such as the Biconjugate Gradient Method.

5.2 Computation of homogenized constitutive parameters

In this section, we give the numerical evaluation of the homogenized constitutive matrix A^H . If we denote A_h^H its approximation matrix in discrete space, the expression of its k^{th} column ($k = 1, \dots, 6$) is given by

$$A_{h,k}^H = \int_Y A(y)(e'_k + \nabla_y \chi_h(y)) dy,$$

The solution of the matrix system (50) allows to obtain the degrees of freedom (χ_h) of the whole nodes in the mesh. By using the expansion given in equation (42), the expression of $A_{h,k}^H$ can be written as

$$A_{h,k}^H = \int_Y \sum_{\{i/a_i \in \Xi\}} x_i^k A(y)(e'_k + \nabla_y \varphi_{a_i}(y)) dy,$$

If we denote

$$g_k(y) = \sum_{\{i/a_i \in \Xi\}} x_i^k A(y)(e'_k + \nabla_y \varphi_{a_i}(y)),$$

The use of an adequate numerical integration allows us to write the k^{th} column of A_h^H in the following form

$$A_{h,k}^H = \sum_{K \in \mathcal{T}_h} \sum_j \omega_j g_k(y_j),$$

where y_j , r and ω_j are, respectively, the integration points, the number of the integration points and the weight associated to j^{th} point.

6. Numerical results

In the following section, we present numerical results concerning the homogenized constitutive parameters of a three dimensions (3D) composite periodic materials. The material studied here is periodic and perforated with cubical inclusions. The homogenized constitutive parameters are computed for different sizes of inclusions.

The permittivity $\bar{\epsilon}$, the permeability $\bar{\mu}$, the two cross-polarization dyadics $\bar{\xi}$ and $\bar{\zeta}$ can be written as

$$\bar{\epsilon} = \epsilon_0 \bar{\epsilon}_r, \quad \bar{\mu}_0 = \mu_0 \bar{\mu}_r, \quad \bar{\xi} = \sqrt{\epsilon_0 \mu_0} \bar{\xi}_r, \quad \bar{\zeta} = \sqrt{\epsilon_0 \mu_0} \bar{\zeta}_r$$

where ϵ_0 and μ_0 are the permittivity and permeability of the vacuum, $\bar{\epsilon}_r$ and $\bar{\mu}_r$ are the relative permittivity and permeability, and $\bar{\xi}_r$ and $\bar{\zeta}_r$ are two relative cross-polarization dyadics.

We choose the value of $\bar{\epsilon}$, $\bar{\mu}$, $\bar{\xi}$ and $\bar{\zeta}$ corresponding to a reciprocal biisotropic. In this case, the parameters $\bar{\xi}$ and $\bar{\zeta}$ are interrelated and can be written in terms of the chirality:

$$\bar{\xi} = -\bar{\zeta} = -i\kappa \sqrt{\epsilon_0 \mu_0} I_3$$

where $\bar{\kappa}$ is the chirality term, is responsible for the reciprocal magnetoelectric phenomena.

$$\begin{aligned}\bar{\bar{\epsilon}}_r(x) &= \begin{cases} 1I_3 & \forall x \in S \\ 40I_3 & \forall x \in Y/\bar{S} \end{cases} \\ \bar{\bar{\mu}}_r(x) &= \begin{cases} 1I_3 & \forall x \in S \\ 30I_3 & \forall x \in Y/\bar{S} \end{cases} \\ \bar{\bar{\kappa}}(x) &= \begin{cases} 0I_3 & \forall x \in S \\ 2I_3 & \forall x \in Y/\bar{S} \end{cases}\end{aligned}$$

where S is a cubical inclusion enclosed inside the reference cell $Y = [0, 1]^3$ in \mathbb{R}^3 as depicted in Fig. 2.

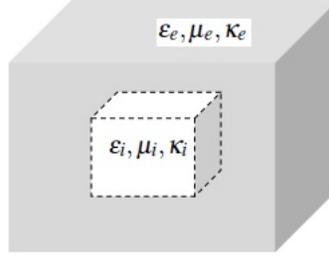


Figure 2. Unit cell composed by two components the host media and the inclusion which characterized respectively by the constitutive parameters $(\epsilon_e, \mu_e, \kappa_e)$ and $(\epsilon_i, \mu_i, \kappa_i)$.

We define the inclusion's volume fraction f as the ratio:

$$f = \frac{|S|}{|Y|}, \quad (51)$$

where $|S|$ denotes the volume of S in \mathbb{R}^3 and denote the area of S in \mathbb{R}^2 . When f equal zero, the microstructure

Table 1. Homogenized relative permittivity, homogenized relative permeability and homogenized chirality as function of the volume fraction f .

h	$f = h^3$	ϵ_r^H	μ_r^H	κ^H
0.000	0.000	40.000	30.000	2.000
0.500	0.125	33.162	24.936	1.645
0.600	0.216	28.671	21.601	1.409
0.700	0.343	23.192	17.532	1.132
0.800	0.512	16.216	12.358	0.774
0.900	0.729	8.866	6.868	0.399
0.950	0.857	5.003	3.987	0.203
0.975	0.926	3.007	2.492	0.106
1.000	1.000	1.000	1.000	0.000

does not contain the inclusions and when f equal 1, the inclusion occupies the whole cell.

The table 1 presents the homogenized relative permittivity ϵ_r^H , the homogenized relative permeability μ_r^H and the homogenized κ^H as function of the volume fraction f . The main difficulty we have met during the computation of the homogenized constitutive parameters is that for each value of f , we had to create a new mesh and re-execute the finite elements program.

7. Conclusion

We have presented the homogenization of the Maxwell equations with periodically oscillating coefficients in the general case. The studied electromagnetic problem are realized in frequency domain and by solving the local problem on the unit cell with boundary conditions. Different aspects are treated in this study. Indeed, the theoretical foundation of the homogenization of Maxwell equations, the numerical analysis of the problem and the numerical results of the homogenized parameters are presented. We note that the used method is the finite element method.

8. Appendix

Theorem 8.1. Let $u^\alpha \in L^2(\Omega; \mathbb{C}^3)$. Suppose that there exists a constant $C > 0$ such that:

$$\|u^\alpha\|_{L^2(\Omega; \mathbb{C}^3)} \leq C \quad \forall \alpha$$

Then a subsequence (still denoted by α) can be extracted from α such that, letting $\alpha \rightarrow 0$

$$\int_{\Omega} u^\alpha(x) \cdot \psi(x, x/\alpha) dx \rightarrow \int_{\Omega} \int_Y u_0(x, y) \cdot \psi(x, y) dy dx$$

for all $\psi \in C_0(\bar{\Omega}; C_{per}(Y; \mathbb{C}^3))$, where $u_0 \in L^2(\Omega; L^2_{per}(Y; \mathbb{C}^3))$. Moreover,

$$\int_{\Omega} u^\alpha(x) \cdot v(x) w(x/\alpha) dx \rightarrow \int_{\Omega} \int_Y u_0(x, y) \cdot v(x) w(y) dy dx$$

for all $v \in C_0(\bar{\Omega}; \mathbb{C}^3)$, and $w \in L^2_{per}(Y)$.

The field u_0 is uniquely expressed in the form

$$u_0(x, y) = u(x) + \tilde{u}_0(x, y)$$

where $\int_{\Omega} \tilde{u}_0(x, y) dy = 0$

Lemma 8.2. Let $f \in H^1_{per}(Y; \mathbb{C}^3)$ and assume that $\nabla_y \times f(y) = 0$. Moreover, assume $\langle f \rangle = 0$. Then there exists a unique function $q \in H^1_{per}/\mathbb{C}$ such that

$$f(y) = \nabla_y q(y)$$

Lemma 8.3. Let $f(y) \in L^1_{per}(Y)$ be a periodic function. There exists a unique solution in H^1_{per}/\mathbb{R} of

$$\begin{cases} -\operatorname{div} A(y) \nabla w(y) = f & \text{in } Y \\ y \rightarrow w(y) & Y\text{-periodic} \end{cases} \quad (52)$$

if and only if $\int_Y f(y) dy = 0$.

Theorem 8.4. Let $u^\alpha \in H(\operatorname{curl}, \Omega)$. Suppose that there exists a constant $C > 0$ such that

$$\|u^\alpha\|_{H(\operatorname{curl}, \Omega)} \leq C \quad \forall \alpha \quad (53)$$

Then a subsequence (still denoted by α) can be extracted from α such that, letting $\alpha \rightarrow 0$

$$u^\alpha \rightarrow u_0 \quad \text{in } L^2(\Omega; \mathbb{C}^3) - \text{weak}$$

and

$$\int_{\Omega} \nabla \times u^\alpha(x) \cdot v(x) w(x/\alpha) dx \rightarrow \int_{\Omega} \int_Y \{\nabla_x \times u_0(x, y) + \nabla_y \times u_1(x, y)\} \cdot v(x) w(y) dy dx$$

for all $v \in C_0(\bar{\Omega})$, and all $w \in L^2_{per}(Y; \mathbb{C}^3)$, where $u_1 \in L^2(\Omega; H_{per}(\operatorname{curl}, Y))$.

Theorem 8.5 (Nguetseng). Let $u^\alpha \in L^2(\Omega)$. Suppose that there exists a constant $C > 0$ such that

$$\|u^\alpha\|_{H(\text{curl}, \Omega)} \leq C \quad \forall \alpha$$

Then a subsequence (still denoted by α) can be extracted from α such that, letting $\alpha \rightarrow 0$

$$\int_{\Omega} u^\alpha(x) \cdot \psi(x/\alpha) dx \rightarrow \int_{\Omega} \int_Y u_0(x, y) \psi(x/\alpha) dx$$

for all $\psi \in C_0(\bar{\Omega}; C_{\text{per}}(Y))$, where $u_0 \in L^2(\Omega; L^2_{\text{per}}(Y))$. Moreover,

$$\int_{\Omega} u^\alpha(x) \cdot v(x) w(x/\alpha) dx \rightarrow \int_{\Omega} \int_Y u_0(x, y) v(x) w(y) dy dx$$

for all $v \in C_0(\bar{\Omega})$, and all $w \in L^2_{\text{per}}(Y)$.

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