

Estimation of Default Risk in CIR++ model simulation

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ABSTRACT

Default risk has always been a matter of importance for financial managers and scholars. In this paper we apply an intensity-based approach for default estimation with a software simulation of the Cox-Ingersoll-Ross model. We analyze the possibilities and effects of a non-linear dependence between economic and financial state variables and the default density, as specified by the theoretical model. Then we perform a test for verifying how simulation techniques can improve the analysis of such complex relations when closed-form solutions are either not available or hard to come by.

KEYWORDS

Default estimation — default density — CIR++ — numerical methods — Monte Carlo — strategic decisions — decision making.

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1. Introduction

Default risk has always been an important element of security assessment for managers, market analysts and investors. Although there are many studies devoted to this topic, recent issues with default probability miscalculations have sparked debates over the accuracy of the methodologies employed and if they are sufficient to cover all aspects of a default event. In this study we will analyze possible applications of the Cox-Ingersoll-Ross model for default estimation. We refer to default as a situation when obligations are not honored by security issuer, thus resulting in a loss. This definition is not explicitly bound to legal requirements and procedures, but reflects the economic implications of refused payments and how they affect security prices and investors. Depending on the issue type and volume it is also possible to witness secondary effects of failing to meet obligations (for example in a case where an entire country defaults or introduces a large percentage haircut), but these effects are not explicitly analyzed here.

Our main goal has been to verify whether a generic CIR++ process can be used to model default event probability and if such a solution is flexible enough for application in different environments – starting with analysis of sovereign debt and ending with applications that target a single company. Such a model has to be able to explain and cope with different event types and a rather complex definition of "default" (contrary to the simplified perception under which the issuer is either fully complying with obligations or completely cutting off all payments) that deals with 2014 ISDA set of credit events like government bail-ins.

As Figure 1. demonstrates, credit default modeling is closely related to market movements and information conveyed in prices. However, as [1] points out information is available in different forms – as historical data

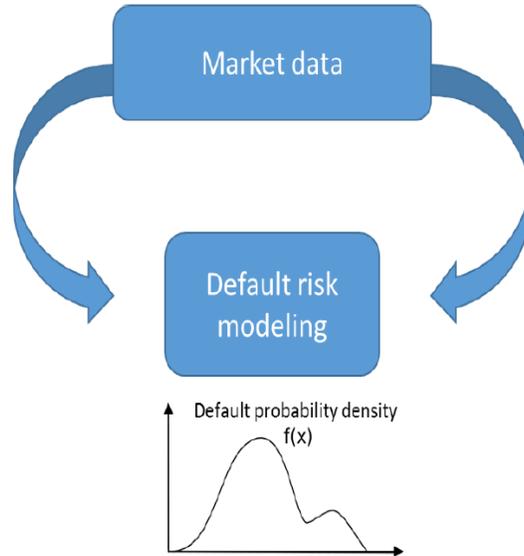


Figure 1. : *Default events and credit risk.*

about observed default events and as implied default probability embedded into required returns. In this paper we attempt to model implied default probability and information on observed default events is used only as a mean to verify the accuracy of the created model. This approach makes it possible to address credit risk assessment and at the same time to analyze if differences between observed yields (typically calculated in excess of risk-free / benchmark return rate) and those predicted by the model are due to deficiency in modeling or should be attributed to other types of risk (for example, liquidity risk).

2. Default Risk

Estimation of default risk can be accomplished using different models, each with its own assumptions and limitations. In general different approaches can be grouped into the following categories:

- Structural default risk estimation approach best known for Merton model [2] and Black-Cox model [3], which put emphasis on corporate bonds and has a strong dependence on option valuation techniques.

Both Merton and Black-Cox models use option pricing theory and the notion that ownership can be represented as having an option to repay the debts and obtain the (positive) difference between company value and debt or go for a default in case the total value is not enough to cover debt payments. In such a way equity (or the difference between company value and debt payments) is considered a call option and depending on the model assumptions may be valued either as a European call (where as in Merton model, the default event can occur at the debt maturity date) or as an American option. In the latter case, as in first passage models, the default event occurs at the very first time the company value is less than or equal to a predefined barrier value. Depending on the model assumption the barrier value may differ from the notional debt amount or total sum of debt payments.

Structural approach allows creating flexible models that are able to reflect actual market behavior but they also have some important drawbacks:

- Default events and procedures for announcing defaults may not happen instantaneously in practice.

Default procedures and default events may differ in timing from pure value calculation due to legal procedures (for example, a company may request legal form of protection and try to restructure or a formal declaration of credit event may have to be announced) which differs from simple structural model assumptions. As [4] points out, structural models can be extended to include state dependence and non-immediate default announcement but that also increases their complexity and requires additional calibration and justification.

- Companies may choose to stay operational even in case of negative equity whenever this is not explicitly prohibited by legislation rules.

There are examples where a company may continue to exist and operate even though it is profitable for the owners to exercise an option and default. For credit risk modeling such cases are not of primary concern because actual decisions can be explained by taking into account that default is just one of the options that company management and owners have. Therefore structural models can be enhanced to handle situations like these by trying to value not a single option, but a portfolio of options instead. Some of them could be related to strategic decisions involving arguments other than the current calculation of equity.

- Default events may be influenced by factors that are not part of the normal market development and that are also not directly related to the economic conditions of the analyzed company/debtor.

A notable example from the recent past is the implementation of bond-buyback programs by central banks (also referred to as quantitative easing rounds) that may act as a real game-rule change that invalidates existing models by simply offering options that did not exist before. Such an example is re-financing debt for issuers in distress at rates that are even lower than the rates of their current debt securities.

- Reduced-form default risk estimation approach which addresses defaults as surprising events with no attempt to explicitly link them to company value or security issuer status.

The important advantage of such an approach is that it does not impose any assumptions on the variables that explain default events and it also does not impose limitations on the factors that trigger them. Structural approaches are by virtue limited because they are built around a finite (and typically small) set of explanatory variables. This argument is developed further by [5] and supported by [6], comparing reduced-form to structural approaches. It should be noted that advantages and disadvantages that we are discussing for the general setup do not necessarily hold for every case. A well-developed structural model with carefully selected variables could outperform a reduced-form based one if actual market development matches the assumptions that have been used when creating the model. Assuming no explicit link between a predefined set of explanatory variables and the probability of default (or expected time of default) is valuable in cases without enough evidence of a significant dependency between explanatory variables and company ability to honour its obligations. However, this flexibility comes at a price - reduced-form models are able to fit available data quite well, but offer little information on how default events can be explained theoretically.

3. CIR++ Model

Mathematical background of reduced-form default models is often closely related to the one used for explaining changes of short term interest rates. Single factor interest rate models assume that current levels of rates (treated as a stochastic variable) determine their future values. In a similar setting, current debtor condition may be considered when determining the probability that obligations will (or will not) be honored in the future.

Within the scope and purpose of this paper this leads to two important consequences:

- results and applications of short term interest rate modeling can be used also for analyzing probability of a default;
- limitations and problems of single factor interest rate models that employ stochastic processes will be also valid for default risk analysis.

Single factor interest rate models can be extended in different ways in order to overcome some of their limitations. For example, one approach is to allow the model parameters to vary over time (which would make it possible to fit the model better to observed changes on the market). For our purposes we are only focusing on two important extensions – the introduction of jumps which can be used to represent ‘non-expected’ events and the mean-reversion property of the employed processes that can be used to describe return to average or ‘fair’ value that eliminates noise and short-term deviations. It should be noted that regardless of the presence or absence of jumps, both structural and reduced-form models are not able to fully cope with truly surprising events. The latter often come with no indication or are triggered by frauds or other activities that have not been reflected in any variables or market expectations.

A commonly used process for describing short term rate evolution which can also be used for default risk analysis is the Cox-Ingersoll-Ross process:

$$dx_t = k(\theta - x_t)dt + \sigma\sqrt{x_t}dW_t \quad (1)$$

It depends on three parameters which determine the mean reversion characteristics of the process:

- k determines how fast the adjustment is;
- θ specifies the long term mean value, that is the value that process reverts to in long term;
- σ is used to describe the volatility of the variable x .

and W_t denotes a Wiener process and represents market risk and changes in the market. The Cox-Ingersoll-Ross process has one important feature that we are going to use - the value of the variable x cannot be negative, and provided that $2k\theta > \sigma^2$ and $x_0 > 0$ it is strictly positive. This can be used to model different investor behavior regarding default risk - that default event is triggered always when available resources for the debtor are equal or less than the amount of required payments. In order to use Cox-Ingersoll-Ross and similar models for analyzing complex systems and running simulations, two questions have to be answered:

- how to calculate model/process parameters - e.g. to calibrate it for a particular environment or problem.

Model calibration is critical for obtaining meaningful results. One of the easiest methods to implement it is to perform discretization of equation 1 and then use available data for small time intervals to calculate parameter values that fit it best (for example by using a regression and ordinary least squares method).

$$dx_{t+\Delta t} = x_t + k(\theta - x_t)\Delta t + \sigma\sqrt{x_t}\epsilon_t \quad (2)$$

The procedure shown by equation 2 is easy to implement and the logic behind it is straightforward. However, it is only usable for a limited number of cases where constant parameter values are able to approximate the actual market data.

- how to simulate the process in order to analyze different scenarios.

[7] has presented in details the advantages and drawbacks of different discretization techniques for CIR processes. In this study we use the discretization approach that allows to simulate CIR process as shown in equation 3.

$$dx_{t+\Delta t} = x_t + k(\theta - x_t)\Delta t + \sigma\sqrt{x_t^+}\sqrt{\Delta t}Z_t \quad (3)$$

where Z_t is a standard normal random variable and the time steps Δt are small. Equation 3 can be derived if we use the characteristics of a Brownian motion:

$$\begin{cases} dx_{t+\Delta t} = x_t + k(\theta - x_t)\Delta t + \sigma\sqrt{x_t^+}(W_{t+\Delta t} - W_t) \\ W_{t+\Delta t} - W_t \sim N(0, \Delta t) \sim \Delta t N(0, 1) \sim \sqrt{\Delta t}Z_t \end{cases} \quad (4)$$

This discretization method is sufficiently fast and accurate for the goals of this study. It can also be used in the context of extended Cox-Ingersoll-Ross (CIR++), defined in equation 4, which links the CIR process to the spot interest rates:

$$\begin{cases} dx_t = k(\theta - x_t)dt + \sigma\sqrt{x_t}dW_t \\ r_t = x_t + \phi_t \end{cases} \quad (5)$$

where r_t is the spot interest rate and ϕ_t is a model-dependent function, which in the case of interest rate estimation depends on the zero coupon bonds market value. We will use this feature during the numerical study, but before that we need to point out that one of the important advantages of CIR++ is that no-arbitrage condition is implicitly included in it. This means that parameter estimation can be continuously performed using currently available market data and that values estimated should also fit very closely to the observed rates of return and market prices.

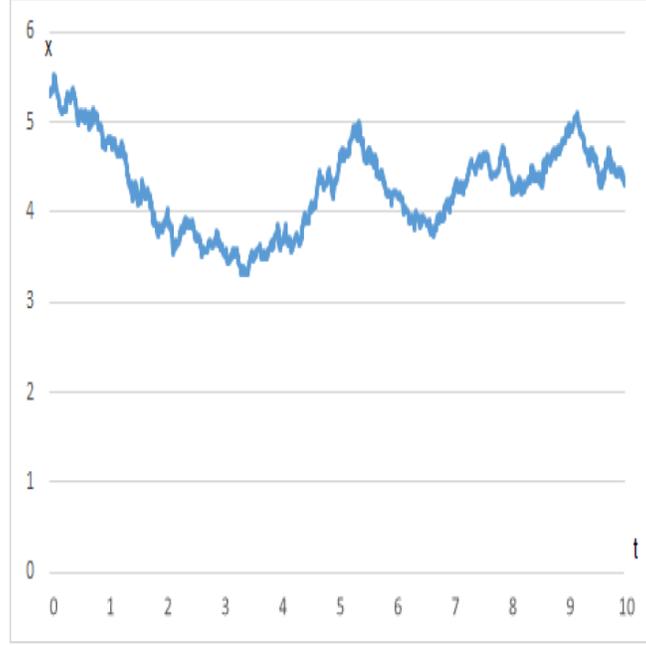


Figure 2. Sample evolution for CIR process generated with discretization method (3) with $x_0 = 5.35\%$, $k = 0.015$, $\theta = 0.0277$, $\sigma = 0.0225$.

4. Default Risk Estimation

Extended CIR process can also be applied to tracking the evolution of default intensity in an equation that is similar to 5:

$$\begin{cases} dy_t = \gamma(\mu - y_t)dt + v\sqrt{y_t}dP_t \\ \lambda_t = y_t + \phi_t \end{cases} \quad (6)$$

Where λ denotes the default intensity and the function ϕ_t is again model-specific defined in a way similar to the usage of CIR++ for interest rate analysis. In this case the parameters are different from 5 but the main properties of the process are the same. Therefore the main factors for the effective use of CIR++ for default analysis are calibration procedure with parameter estimation (γ, μ and v) and choice of a suitable function ϕ_t . For the numerical study we have made an attempt to calibrate CIR++ model for default intensity based on observed market quotations of credit default swaps.

We assume that the present value of cash flows associated with a CDS contract - protection leg (contingent payments in case of a default) and fixed payments for ensuring this protection, are equal thus making the CDS net value 0 at the current moment. Equation 7 denotes the present value of these cash flows in case of discrete time periods:

$$\begin{cases} PV(1) = \sum_{t=1}^N e^{-rt}(1 - DP_t)P_t + \sum_{t=1}^N e^{-rt}(DP_t - DP_{t-1})\frac{P_t}{2} \\ PV(2) = (1 - R)\sum_{t=1}^N e^{-rt}(DP_t - DP_{t-1}) \end{cases} \quad (7)$$

Where DP_t is the default probability at a time t and $1 - DP_t$ is the "survival" probability which means that no default event has happened. P_t is the payment associated with period t - typically CDS premium payments are done every 3 months or 4 times per year. R is the recovery rate - that is the percentage of notional amount that is paid in case of a default. In case the full amount is paid upon default, then the recovery rate would be 0 (meaning that $1 - R$ refers to the full notional amount specified in the CDS contract). We have also assumed that in case a default event occurs between two payment dates we can approximate it by using an average of the payment amount. Using equations 7 and the assumption that $PV(1) = PV(2)$ in case of a zero net value CDS contract we

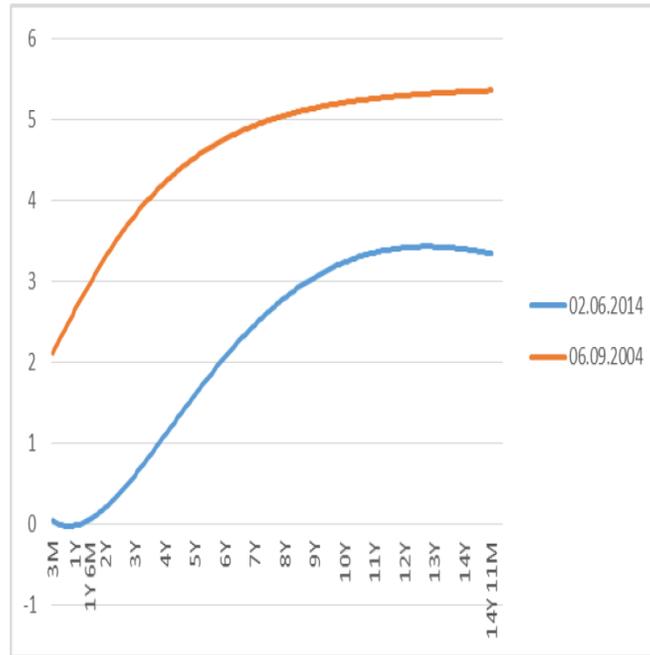


Figure 3. Sample yield curves used for calibrating the model.

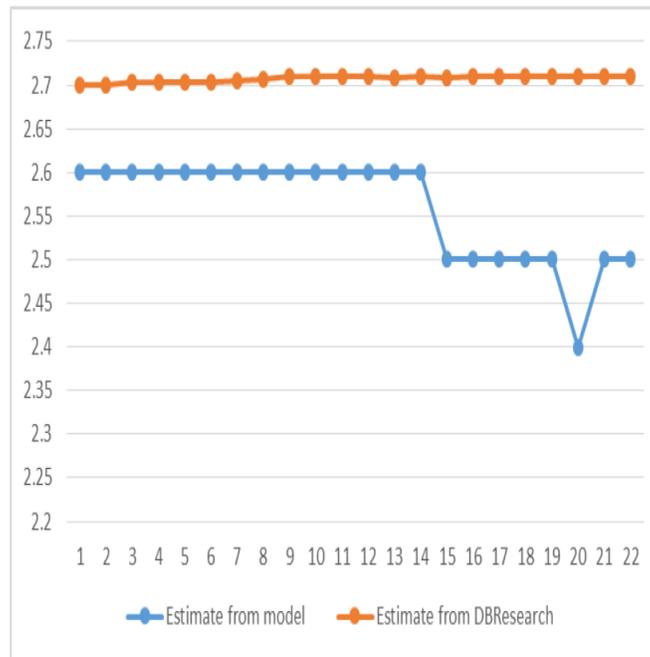


Figure 4. STesting model results against estimates from Deutsche Bank Research with $x_0 = 0.027$, $k = 1.255$, $\theta = 0.029$, $\sigma = 0.027$.

can calibrate CIR++ process parameters for short rate and default intensity - using respectively the data for yield curve and available CDS prices on the market. Once calibrated, the output of the models can be used to analyze default probability (after transforming the default intensity) and default risk.

Figure 4 demonstrates that when the model is tested against relatively stable CDS data it returns higher than benchmark changes in the expected probability of default. This effect can be flattened out if calibration is done

with values spreading over a larger period of time and if we take into account that observed CDS prices reflect not only risk of default but also counterparty risk and liquidity risk.

Condition	Description
Information on CDS quotes used to calibrate the model.	Credit default swaps have been selected as a measure for risk as they are typically used in addition to (or even instead of) credit ratings and also they are usable for targeting both company and sovereign debt. We have used CDS spread data from [8]. Expected probability of default with 60% recovery rate has been used as a benchmark.
Information on interest rates is taken from Euro area yield curve [9]	Information on instantaneous forward rates on AAA bonds with maturity from 3 months to 15 years has been used to build yield curves. Data from September 2004 till June 2014 has been used in order to make sure that different yield curve forms will be included. This is also shown on Figure 3, which plots two sample curves from the data set.

Table 1. Description of calculations and conducted experiment.

5. Conclusion

We have demonstrated a way to use CIR++ processes for default risk analysis, employing CDS spreads as a proxy of default event risk. We have used a rather simple way of representing CDS pricing process, where present value of contingent payments (depending on the probability of a default), calculated over discrete periods of time is equal to the present value of protection payments. This approach allows to build simple and easy to implement models for default risk assessment. It is also possible to improve the analysis in several ways. First, the CIR++ process is built in a way that precludes very sharp changes in interest rates cite10, which does not always fit with the actual market developments. Although large changes as well as sudden jumps in probability of default (typically due to revealing unknown information) may be rare, they do happen and then the selected base process is not able to match the reality. Second, the discretization used for representing CDS valuation process could be enhanced to avoid using average values in the second term of equation 7. Third, the CDS spread reflects several types of risk, most notably counterparty risk and not just default risk. This requires modification of the model and its additional calibration, which falls beyond the scope of this paper.

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