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Error estimation of electromagnetic properties in periodic composite materials

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ABSTRACT
This paper addresses the a priori error estimation of the homogenized constitutive parameters (HCPs), the macroscopic field and the limit electromagnetic field in 3D periodic structure. The HCPs and the macroscopic field are approximated respectively by using the Lagrange and the first order Nédélec conforming finite element method. The approximation of limit field is derived from those of HCPs and macroscopic field. The optimality of the convergence is obtained for these electromagnetic quantities and the theoretical results of this work are reinforced by some numerical ones.

KEYWORDS
Maxwell equations — Homogenization — Periodic composite material — Finite element approximation — Error estimation.

1. Introduction

The composite periodic structures have been investigated extensively in different branches of engineering such as electromagnetism, heat conduction, elastic deformation, porous media, acoustics [1]-[11]. In electromagnetism, the most famous types of these periodic structures are the metamaterials that have great potential applications and include sensor detection and infrastructure monitoring, remote aerospace applications, public safety, smart solar power management, radomes, the thin film capacitor design for the use in a new generation of memory devices, high-frequency battlefield communication and lenses for high-gain antennas, improving ultrasonic sensors, and even shielding structures from earthquakes [12]-[18].

Artificial materials are synthesized to obtain the desired electromagnetic properties which can not be found in the nature. They usually gain their properties from the composition of the structure, the distribution and shape of inclusions. The homogenization is one of the most used method to characterize and model these materials. When the period of this composite material is small compared to the size of the studied structure, the homogenization processus allows establishing the homogenized electromagnetic properties by taking account the properties of the different heterogeneities. This means that the heterogeneous material is replaced by an homogeneous fictitious one whose global characteristics are a good approximation of the initial material. There are numerous approaches have been proposed providing the homogenized constitutive parameters (HCPs) of the Maxwell's equations both in frequency domain and in time domain. Barbatis [19] and Wellander [20] used the concept of two-scale homogenization technique, and Bossavit [21] employed the classical multi-scale homogenization technique giving a new approach based upon the periodic unfolding method. Generally, the boundary conditions used are the perfect conductor walls [22]-[24] or penetrable boundary conditions [20].

Assume Ω is a smooth and bounded domain in $\mathbb{R}^3$ with boundary $\partial\Omega$. Moreover, assume that the material in Ω is Y^α-periodic ($Y^\alpha = \alpha Y$), where $Y = (0,1)^3$ is the unit cube in $\mathbb{R}^3$. In the case of the anisotropic materials, the homogenized permittivity, ε^H, and permeability, μ^H, are described by their columns [20]-[21].

$$\varepsilon_k^H = \int_Y \varepsilon(y)\cdot(e_k + \nabla w_k^\varepsilon(y))dy \tag{1}$$

$$\mu_k^H = \int_Y \mu(y)\cdot(e_k + \nabla w_k^\mu(y))dy \tag{2}$$

where e_k is the k-th canonical vector basis of $\mathbb{R}^3$, $\varepsilon(y)$ is the permittivity and $\mu(y)$ is the permeability of materials in Y. The sub-correctors w_k^ε and w_k^μ for $k = 1,2,3$ are solutions of the local problems for all v in $H^1_{per}(Y)$

$$\int_Y \nabla_y v(y)\cdot\varepsilon(y)\cdot(\nabla w_k^\varepsilon(y) + e_k)\ dy = 0, \tag{3}$$

$$\int_Y \nabla_y v(y)\cdot\mu(y)\cdot\left(\nabla w_k^\mu(y) + e_k\right)\ dy = 0. \tag{4}$$

In addition, the behaviour of the electromagnetic field limit is also treated. In a α-periodic material, the electromagnetic fields satisfy the Maxwell equations in Ω. They depend on the period of the material. Therefore, all fields are indexed by the periodicity α, namely, $(\mathbf{E}^\alpha, \mathbf{H}^\alpha)$. These fields converge weakly in $H(\mathbf{curl},\Omega)\times H(\mathbf{curl},\Omega)$ to the macroscopic field $(\mathbf{E}^m,\mathbf{H}^m) \in H(\mathbf{curl},\Omega)\times H(\mathbf{curl},\Omega)$ [19, 21], where

$$H(\mathbf{curl},\Omega) = \{\mathbf{v}\in L^2(\Omega), \quad \mathbf{curl}\ \mathbf{v}\in L^2(\Omega)\}. \tag{5}$$

It was shown in [19, 20] that the use of the concept of two-scale convergence, the field $(\mathbf{E}^\alpha, \mathbf{H}^\alpha)$ converges to $(\mathbf{E}^l, \mathbf{H}^l)$ which expressed by

$$\mathbf{E}^\alpha \overset{2-s}{\rightharpoonup} \mathbf{E}^l(x,y) = \mathbf{E}^m(x) + \mathbf{E}^c(x,y) \tag{6}$$

$$\mathbf{H}^\alpha \overset{2-s}{\rightharpoonup} \mathbf{H}^l(x,y) = \mathbf{H}^m(x) + \mathbf{H}^c(x,y) \tag{7}$$

where $\overset{2-s}{\rightharpoonup}$ denotes the two-scale limit, and $\mathbf{E}^c(x,y)$ and $\mathbf{H}^c(x,y)$ are expressed as

$$\mathbf{E}^c(x,y) = \nabla_y\phi(x,y), \qquad \mathbf{H}^c(x,y) = \nabla_y\psi(x,y) \tag{8}$$

The fields $(\mathbf{E}^c,\mathbf{H}^c)$ and $(\mathbf{E}^l,\mathbf{H}^l)$ are, respectively, the corrector and the limit electromagnetic fields. The functions ϕ and ψ in equations (8) contain the information of the behaviour of the fields on the microscale. A separation of variables arguments implies that these terms can be written as [20]

$$\mathbf{E}^c(x,y) = \sum_{k=1}^{3} \nabla_y w_k^\varepsilon(y)\mathbf{E}_k^m(x), \tag{9}$$

$$\mathbf{H}^c(x,y) = \sum_{k=1}^{3} \nabla_y w_k^\mu(y)\mathbf{H}_k^m(x) \tag{10}$$

By utilizing the periodic unfolding operator $\mathscr{T}_\alpha$, we have the following limit [21]

$$\mathscr{T}_\alpha(\mathbf{E}^\alpha)(x,y) \to \mathbf{E}^l(x,y) \qquad \text{strongly in} \quad L^2(\Omega\times Y;\mathbb{R}^3), \tag{11}$$

$$\mathscr{T}_\alpha(\mathbf{H}^\alpha)(x,y) \to \mathbf{H}^l(x,y) \qquad \text{strongly in} \quad L^2(\Omega\times Y;\mathbb{R}^3), \tag{12}$$

where x is macroscopic variable and y is the the microscopic variable. The macroscopic field $(\mathbf{E}^m,\mathbf{H}^m)$ is independent of the variable y. These fields satisfy the Maxwell equations characterized by the HCPs (ε^H,μ^H). Using the constitutive relations ($\mathbf{D}^m = \varepsilon^H\mathbf{E}^m$, $\mathbf{B}^m = \mu^H\mathbf{H}^m$), the time-harmonic Maxwell equations are given in free space by

$$\mathbf{curl}\ \mathbf{E}^m(x) = i\omega\mu^H\mathbf{H}^m(x), \tag{13}$$

$$\mathbf{curl}\ \mathbf{H}^m(x) = -i\omega\varepsilon^H\mathbf{E}^m(x). \tag{14}$$

The numerical results of the homogenized constitutive parameters, the macroscopic fields and the corrector fields are presented for some classes of the electromagnetic materials as the isotropic, anisotropic, chiral and bi-anisotropic materials [25]-[32]. This paper is devoted to error estimation of the HCPs (ε^H, μ^H), the macroscopic electromagnetic field $(\mathbf{E}^m, \mathbf{H}^m)$ and the limit electromagnetic field $(\mathbf{E}^l, \mathbf{H}^l)$ when we employ the finite element method. Usually, the electromagnetic field solution of Maxwell equations with exact coefficients (ε, μ) is approximated by the use of the first order Nédélec conforming finite element method. The optimal error made is the order of $O(h^{\min(s,1)})$ [33]-[34], where s is the regularity parameter of the exact macroscopic field ($(\mathbf{E}^m, \mathbf{H}^m) \in W^{s,2} \times W^{s,2}$). Here, the situation is different, the difficulty is to take account the approximated coefficients (ε^H, μ^H). Our technique to counter this problem consists to use the Strang lemma. However, this latter requires some conditions which are not evident to satisfy. The main result in this case consists of showing that we still obtain the optimal convergence when the approximated coefficients (ε^H, μ^H) have the optimal convergence too. In addition, we prove the optimality of these coefficients, which is related to the optimal approximation of the sub-correctors (w^ε, w^μ) in Eqs (3-4). The difficulty to control the error of the corrector field (9-10) is the product of the two approximated functions. However, we still have the optimal approximation. The outline of this paper is as follows. In Section 2, the algorithm, containing the different steps to obtain diverse electromagnetic quantities, is presented. Section 3 is devoted to approximate the continuous problem given in Section 2 using finite element discretization. The error estimate of the HCPs, the macroscopic and the limit field are established in Section 4. In the last Section, we provide some numerical experiments to validate the theoretical results.

2. Algorithm

The limit field $(\mathbf{E}^l, \mathbf{H}^l)$ in periodic composite material, obtained using two-scale convergence or unfolding method, is given by the macroscopic field $(\mathbf{E}^m, \mathbf{H}^m)$ and the corrector field $(\mathbf{E}^c, \mathbf{H}^c)$. In this section, we describe the different steps in order to give the error estimation of the different electromagnetic quantities. The algorithm presented here contains four steps. The first step is devoted to evaluate the sub-correctors w^ε and w^μ solutions of the local problem (3-4), the HCPs (ε^H, μ^H) in equation (1-2) are presented in second step. The third one consists to give the macroscopic field. In the last step, we present the corrector field and we deduce the limit field.

2.1 Step I: Evaluation of w^ε and w^μ solutions of local problem

Since the HCPs are given as function of local terms w_k^ε and w_k^μ for $k = 1,2,3$, we start by solving the local problem in $V = H^1_{per}(Y)/\mathbb{C}$. The associate problems of w^ε and w^μ have the same form and they can be written in the same way by w^β where β is either ε or μ.

$$\text{Find} \quad w_k^\beta \in V \quad \text{such that} \quad a(w_k^\beta, v) = l(v), \quad \forall v \in V, \quad k = 1,2,3. \tag{15}$$

where

$$a(w_k^\beta, v) = \int_Y \nabla_y v(y) \cdot \beta(y) \cdot \nabla w_k^\beta(y) dy \tag{16}$$

$$l(v) = -\int_Y \nabla_y w(y) \cdot \beta(y) \cdot e_k \, dy \tag{17}$$

Here, we assume that $\beta(y)$ ($\beta = \varepsilon$ or μ) is in $L^\infty(\Omega)$ and there exists $c_1 > 0$ and $c_2 > 0$ such that β satisfies the following bounds

$$c_1 |z|^2 \geq \sum_{i,j=1}^{3} \beta_{i,j}(y) z_i z_j \geq c_2 |z|^2, \qquad \forall z \in \mathbb{R}^3$$

It is easy to check that there exists a unique solution of the problem (15) up to a constant.

2.2 Step II: Evaluation of HCPs ε^H and μ^H

The HCPs (ε^H, μ^H) are given by the permittivity $\varepsilon(y)$ and the permeability $\mu(y)$ of material in unit cell Y, and the terms w^ε, w^μ solutions of local problem (15). These parameters are present in the next step devoted to evaluate the macroscopic field $(\mathbf{E}^m, \mathbf{H}^m)$.

2.3 Step III: Evaluation of the macroscopic field $(\mathbf{E}^m, \mathbf{H}^m)$

This part deals with the macroscopic field $(\mathbf{E}^m, \mathbf{H}^m)$, which is solution of Maxwell equations in the bounded material Ω characterized by the HCPs (ε^H, μ^H). The equations (13-14) are usually reformulated in term of the macroscopic electric field $\mathbf{E}^m$ or in term of the macroscopic magnetic field $\mathbf{H}^m$. In the following, we will analyze only the electric field. The magnetic analysis can be obtained by the same way, and satisfies the same mean results. By re-writing this system and taking a boundary conditions, we obtain the following problem

$$\left\{\begin{array}{lcll} \mathbf{curl}([\mu^H]^{-1}\mathbf{curl}\,\mathbf{E}^m) - \omega^2\varepsilon^H\mathbf{E}^m & = & 0 & \text{in} \quad \Omega, \\ \mathbf{E}^m \times \mathbf{n} & = & 0 & \text{on} \quad \partial\Omega_D, \\ \mathbf{curl}\,\mathbf{E}^m \times \mathbf{n} & = & \mathbf{J} & \text{on} \quad \partial\Omega_N. \end{array}\right. \tag{18}$$

where $\mathbf{n}$ is the unit outgoing normal vector and the boundary $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$, with $\partial\Omega_D \cap \partial\Omega_N = \emptyset$. On the part $\partial\Omega_D$ and $\partial\Omega_N$, we impose respectively the perfect conductor and the Neumann condition boundary.

We introduce the following space

$$V \quad = \quad \{\mathbf{v} \in H(\mathbf{curl}, \Omega) \mid (\mathbf{v} \times \mathbf{n}) = 0 \text{ on } \Omega_D \text{ and } \mathbf{v} \times \mathbf{n} \in (L^2(\partial\Omega_N))^3\}. \tag{19}$$

By multiplying the first equation of (18) by a test function in V and taking the boundary condition, the associated variational formulation of (18) reads

$$\text{Find} \quad \mathbf{E}^m \in V \quad \text{such that} \quad a(\mathbf{E}^m, \mathbf{E}') = l(\mathbf{E}'), \quad \forall \mathbf{E}' \in V, \tag{20}$$

where

$$a(\mathbf{E}^m, \mathbf{E}') \quad = \quad \int_\Omega ([\mu^H]^{-1}\mathbf{curl}\,\mathbf{E}^m \cdot \mathbf{curl}\,\mathbf{E}' - \omega^2\varepsilon^H\mathbf{E}^m \cdot \mathbf{E}')\, dx, \tag{21}$$

$$l(\mathbf{E}') \quad = \quad \int_{\partial\Omega_N} [\mu^H]^{-1}\mathbf{J} \cdot \mathbf{E}'\, dx. \tag{22}$$

If $\omega = 0$, the problem (20) has a unique solution using the Lax-Milgram lemma. Otherwise, the presence of the term $-\int_\Omega \omega^2\varepsilon^H\mathbf{E}^m \cdot \mathbf{E}'\, dx$ in the right-hand side of (21) means that the right-hand side is not a coercive sesquilinear form. To counter this problem, we assume the following hypotheses:

(1) The coefficients ε and μ in (20) are piecewise smooth.

(2) The domain Ω may be decomposed into P subdomains such that

- $\overline{\Omega} = \bigcup_{p=1}^{P} \overline{\Omega}_p$ and $\Omega_p \bigcap \Omega_q = \emptyset$, if $p \neq q$;
- Each subdomain Ω_p, $p = 1, ..., P$, is connected and has a Lipschitz boundary;

(3) The coefficient μ is constant on each subdomain;

(4) The coefficient ε is assumed to have the following properties:

- The restriction of ε to Ω_p is a function in $H^3(\Omega_p)$,
- There is a constant $c > 0$ such that for each $p, p = 1, ..., P$, and $\Im(\varepsilon) \geq c$ on Ω_p, where $\Im$ is the imaginary part.

According to Monk [34], we have the following theorem

Theorem 2.1. *Under these assumptions, there is at most one solution $\mathbf{E}^m$ to the problem (20).*

2.4 Step IV: Limit field $(\mathbf{E}^l, \mathbf{H}^l)$

The computation of the terms $w_k^\varepsilon \in H^1_{per}(Y)/\mathbb{C}$ solution of the local problem (15) in Y (Step I) and the macroscopic electric field $\mathbf{E}^m \in V$ solution of the problem (20) (Step III), allows us to evaluate the corrector electric field $\mathbf{E}^c$. We can write

$$\mathbf{E}^c(x, y) = \sum_{k=1}^{3} \nabla_y w_k^\varepsilon(y)\mathbf{E}_k^m(x) \tag{23}$$

Finally, the limit of the electric field $\mathbf{E}^l$ is given by (6):

$$\mathbf{E}^l(x, y) \quad = \quad \mathbf{E}^m(x) + \mathbf{E}^c(x, y) \tag{24}$$

The algorithm presented in this section contains four steps. The organization of this algorithm is respected in the next sections (3 and 4). Section 3 is devoted to finite element discretization of continuous problem of each step and Section 4 deals with the error estimate of the different quantities when we use finite element method.

3. Finite element discretization

In this section, we give the finite element discretization of the different continuous problems presented in previous section. We will also have consider four steps. In the first step, we present the discretization of the terms w^{ε} and w^{μ} by using first order Lagrange conforming finite element method. The second step consists to update the HCPs (ε^H,μ^H). In the third step, the Nédélec elements are employed to approximate the macroscopic field $\mathbf{E}^m$. The update of the limit field, $\mathbf{E}^l$, is presented in the last step.

Let $\mathscr{T}_{\Omega,h}$ and $\mathscr{T}_{Y,h}$ be , respectively, a family of triangulations of $\overline{\Omega}$ and $\overline{Y}$ by means of a mesh composed of tetrahedra K such that

$$\overline{\Omega} = \bigcup_{K\in\mathscr{T}_{\Omega,h}} K \qquad , \quad \overline{Y} = \bigcup_{K\in\mathscr{T}_{Y,h}} K$$

where $h = \max\{\text{diam}(K), K \in \mathscr{T}_{Z,h}\}$ where $Z = Y$ or Ω. We assume that the mesh is regular in the sense of Ciarlet. We note that to obtain a periodic mesh of Y, the discretization of the opposite face of Y is identical. When Y contains a concentric isotropic inclusion, we start by meshing one-eighth of Y, the other seven-eighth of Y are obtained by using the symmetry in three directions. Finally, we obtain the mesh of the Y by grouping the eight subdomains.

The reference element is defined to be the tetrahedron $\hat{K}$ with vertices $\hat{\mathbf{a}}_1,...,\hat{\mathbf{a}}_4$ given by $\hat{\mathbf{a}}_1 = (0,0,0)^T$, $\hat{\mathbf{a}}_2 = (1,0,0)^T$, $\hat{\mathbf{a}}_3 = (0,1,0)^T$ and $\hat{\mathbf{a}}_4 = (0,0,1)^T$. Any target element $K \in \mathscr{T}_{Z,h}$ can be obtained by mapping $\hat{K}$ using an affine map. By this we mean that for any $K \in \mathscr{T}_{Z,h}$ there is a map $\mathbf{F}_K : \hat{K} \to K$, defined by:

$$\mathbf{F}_K(\hat{\mathbf{x}}) = B_K\hat{\mathbf{x}} + \mathbf{b}_K$$

such that $K = \mathbf{F}_K(\hat{K})$ where B_K is a non-singular 3×3 matrix, and $\mathbf{b}_K$ is a vector. The non-singularity of B_K is a result of the fact that we assumed that K has a non-empty interior since the volume of K is $|\det(B_K)|/6$. If K has vertices $\mathbf{a}_1$,..., $\mathbf{a}_4$ and if we choose $\mathbf{F}_K$ to satisfy $\mathbf{F}_K(\hat{\mathbf{a}}_i) = \mathbf{a}_i$ for $1 \le i \le 4$ then it is easy to compute B_K and $\mathbf{b}_K$, such that $\mathbf{b}_K = \mathbf{a}_1$ and B_K is the matrix where j^{th} column is given by $\mathbf{a}_{j+1} - \mathbf{a}_1$.

3.1 Step I : Discretization of w^{β} $(\beta = \varepsilon, \mu)$

In this part, we give an approximation of the continuous problem (15) expressed in Y. To this end, we use the conformal Lagrange finite elements. Now, we introduce the following space

$$U_h = \{u_h \in \mathscr{C}^0(Y) | \quad u_{h|\Gamma_D} = 0 \text{ and } u_h o F_K^{-1} \in \mathscr{P}_K(\hat{K}) \quad \forall K \in \mathscr{T}_h\},$$

If we denote by w_h^{β} the approximation of w^{β} in U_h, the associated discrete variational problem is expressed as follows

$$\begin{aligned} &\text{Find} \quad w_{k,h}^{\beta} \in U_h \quad \text{such that} \\ &a_h(w_{k,h}^{\beta}, u_h) = l_h(u_h), \quad \forall u_h \in U_h, \quad k = 1,2,3. \end{aligned} \tag{25}$$

where

$$a_h(w_{k,h}^{\beta}, u_h) = \int_Y \nabla_y u_h(y) \cdot \beta(y) \cdot \nabla w_{k,h}^{\beta}(y)\, dy, \quad k = 1,2,3. \tag{26}$$

$$l_h(u) = -\int_Y \nabla_y u_h(y) \cdot \beta(y) \cdot e_k\, dy, \quad k = 1,2,3. \tag{27}$$

Due to the existence and the uniqueness of the solution of the continuous problem (15) and the use of the conformal approximation, the approximate problem (25) has also a unique solution.

3.2 Step II : Update of HCPs (ε^H, μ^H)

From the previous step, the terms $w_{k,h}^{\beta}$ $(\beta = \varepsilon, \mu)$ are the approximations of w_k^{β}. They are consequently attached with an error which is reproduced in the HCPs (ε^H, μ^H). Let us then denote the approximated HCPs by $(\varepsilon_h^H, \mu_h^H)$ which are expressed as

$$\varepsilon_{k,h}^H = \int_Y \varepsilon(y)(e_k + \nabla w_{k,h}^{\varepsilon}(y))dy \tag{28}$$

$$\mu_{k,h}^H = \int_Y \mu(y)(e_k + \nabla w_{k,h}^{\mu}(y))dy \tag{29}$$

3.3 Step III : Discretization of the macroscopic field $\mathbf{E}^m$

In this part, we give the approximation of the continuous problem (20) characterized by the solution $\mathbf{E}^m$. For this, We use the $H(\mathbf{curl},\Omega)$ conformal finite element of Nédélec and we introduce the following finite dimensional space

$$V_h = \{\mathbf{v}_h \in H(\mathbf{curl},\Omega) \quad | \quad B_K^T(\mathbf{v}_{h|K} o \mathbf{F}_K) \in \mathscr{P}_n(\hat{K}), \quad \forall K \in \mathscr{T}_h\}$$

where $\mathscr{P}_n$ denotes a family of Nédélec type finite element polynomials of degree n. The basis functions $\mathbf{w}_{ij}$ of the space $\mathscr{P}_n$ that will be used later in the definition of the basis for V_h. Given a target tetrahedron K, let $\mathbf{r}_j(j = 1,..,4)$ be the position vectors of its vertices and $\lambda_j(\mathbf{r})$ be the barycentric coordinate of the point $P \in K$ with respect to the vertex j. It is clear that $\lambda_j(\mathbf{r})$ is a linear function in the tetrahedron with $\lambda_j(\mathbf{r}_l) = \delta_{jl}$ $(j,l \in \{1,2,3,4\})$. The vector basis function corresponding to an edge e_{ij} going from $\mathbf{r}_i$ to $\mathbf{r}_j$ is given by

$$\mathbf{w}_{ij}(\mathbf{r}) = \lambda_i(\mathbf{r})\mathrm{grad}\lambda_j(\mathbf{r}) - \lambda_j(\mathbf{r})\mathrm{grad}\lambda_i(\mathbf{r})$$

The interpolating function $\mathbf{u}_h$ on K for vectorial state $\mathbf{u} \in (\mathscr{C}^0(\overline{K}))^3$ has the following form

$$\mathbf{u}_h = \sum_{i=1}^{3}\sum_{j>i}^{4} \mathbf{w}_{ij}\alpha_{ij} \qquad \text{with} \qquad \alpha_{ij} = \int_{e_{ij}} \mathbf{u}_h.\mathbf{dl}$$

where $\mathbf{dl}$ is the unit vector tangent displacement on the edge e_{ij}. Usually, the basis functions are expressed on the reference tetrahedron $\hat{K}$ and mapped to target tetrahedron K of the mesh.

In order to obtain the discret problem by taking into account the boundary conditions, we define the approximated finite space given as

$$X_h = \{\mathbf{v}_h \in V_h \quad | \quad \mathbf{v}_h \times \mathbf{n} = 0 \text{ on } \Gamma_D\}.$$

Finally, the discrete problem of the continuous problem (20) can be read by:

$$\text{Find} \quad \mathbf{E}_h^m \in X_h \quad \text{such that} \quad a_h(\mathbf{E}_h^m, \mathbf{E}_h') = l_h(\mathbf{E}_h'), \quad \forall \mathbf{E}_h' \in X_h \tag{30}$$

where

$$a_h(\mathbf{E}_h^m, \mathbf{E}_h') = \int_\Omega ([\mu_h^H]^{-1}\mathbf{curl}\,\mathbf{E}_h^m \cdot \mathbf{curl}\,\mathbf{E}_h' - \omega^2 \varepsilon_h^H \mathbf{E}_h^m \cdot \mathbf{E}_h')\, dx, \tag{31}$$

$$l_h(\mathbf{E}_h') = \int_{\partial\Omega_N} [\mu_h^H]^{-1}\mathbf{J} \cdot \mathbf{E}_h'\, ds \tag{32}$$

The problem (30) has a unique solution, thanks to the use of the conformal approximation and its continuous problem also has a unique solution. A direct proof can be established using the Strang lemma given in section §4.3.

3.4 Step IV : Update of the limit electric field $\mathbf{E}^l$

We denote by $\mathbf{E}_h^l$ and $\mathbf{E}_h^c$ the approximations of $\mathbf{E}^l$ and $\mathbf{E}^c$ in the finite discret space. The expressions of $\mathbf{E}_h^l$ and $\mathbf{E}_h^c$ are given as function of w_h^β and $\mathbf{E}_h^m$, respectively, the approximation of w^β and $\mathbf{E}^m$.

$$\begin{aligned} \mathbf{E}_h^c(x,y) &= \sum_{k=1}^{3} \nabla_y w_{k,h}^{\varepsilon}(y)\mathbf{E}_{k,h}^m(x) \\ \mathbf{E}_h^l(x) &= \mathbf{E}_h^m(x) + \mathbf{E}_h^c(x,y) \end{aligned}$$

In this section, we presented the finite element approximation of the different electromagnetic quantities. In the next section we analyze the a priori error estimate of these quantities.

4. A priori error estimate

In this section, we investigate the a priori error estimate of the previously defined algorithm, computed using conformal finite element method. The strategy used here is to seek for an optimal error estimate at each level (or step) of the algorithm. As mentioned before; four steps are to be considered.

4.1 Step I: A priori error estimation on w^β, $\beta = \varepsilon, \mu$

For the sake of simplicity in notation, we are going to write w^β instead of w_k^β which represents the $k-$th column exact solution. This following result shows that the optimal error estimate is achieved when we use Lagrange finite element method to compute w^β.

Proposition 4.1. *Let w^β be the solution of problem (15) and w_h^β the solution of problem (25). Under the assumption of $\beta(y)$, there exists a constant C not depending on w^β such that*

$$\|w^\beta - w_h^\beta\|_{1,Y} \leq C \inf_{w \in U_h} \|w^\beta - w\|_{1,Y}. \tag{33}$$

Proof. This result is a consequence of Cea's lemma [35]. □

4.2 Step II: A priori error estimation on ε^H and μ^H

Let us recall that ε^H and μ^H are just particular average of ε and μ, with weight depending linearly on first derivatives of w^ε and w^μ. Consequently, the error made in evaluating either ε^H or μ^H depends only in the used quadrature and the error made on evaluating w^β ($\beta = \varepsilon, \mu$). Here we suppose that the used quadrature are rich enough to neglect the error due to this quadrature, more specially in our case where ε and μ are constant per element. Now, we give a standard result which will be applied in estimating either ε^H or μ^H. Due to the Cauchy-Schwartz inequality, one can demonstrate the following lemma.

Lemma 4.2. *Let Y be an open bounded subset of $\mathbb{R}^3$, $\sigma \in L^\infty(Y)$ and $\psi_1, \psi_2 \in H^1(Y)$. Assume χ_1 and χ_2 be defined by*

$$\chi_1 = \int_Y \sigma(y) \nabla \psi_1(y)\, dy, \quad \chi_2 = \int_Y \sigma(y) \nabla \psi_2(y)\, dy \tag{34}$$

then

$$|\chi_1 - \chi_2| \leq |Y| \|\sigma\|_{\infty,Y} \|\psi_1 - \psi_2\|_{1,Y}. \tag{35}$$

where $|Y|$ denotes the Lebesgue measure of the bounded open subset Y.

The application of this lemma allows to control the error between (ε^H, μ^H) and $(\varepsilon_h^H, \mu_h^H)$ as presented in the following proposition.

Proposition 4.3. *Let ε_h^H and μ_h^H be an approximated value of ε^H and μ^H, respectively, obtained by replacing w^ε (resp. w^μ) in (1) (resp. (2)) by their approximations given by (25). Then there exists a constant C depending only on Y, ε, and μ such that*

$$|\varepsilon^H - \varepsilon_h^H| \leq C \|w^\varepsilon - w_h^\varepsilon\|_{1,Y}, \tag{36}$$

$$|\mu^H - \mu_h^H| \leq C \|w^\mu - w_h^\mu\|_{1,Y}. \tag{37}$$

Combining this result with the previous one (Proposition 4.1), we can obtain the error made when evaluating ε^H and μ^H. This error will still be optimal with respect to the mesh size.

4.3 Step III: A priori error estimation on the macroscopic field $\mathbf{E}^m$

The problem satisfied by $\mathbf{E}^m$, has been studied by many books see a review for instance in [34], but here the situation is different since the parameters in equation (20) are only approximated ones. Therefore, this induces a particular error in the approximation of $\mathbf{E}^m$. The way to account this error is to use the Strang lemma. As one can imagine, due to possible semi-definite property of the bilinear form, it will not be easy to show that our problem satisfy this property. But thanks to recent result of Bramble [33] we will circumvent this as shown in Lemma 4.5.

Proposition 4.4 (Strang)**.** *Let X_h be a Hilbert space and $a_h(\cdot,\cdot)$ be a bilinear form defined on X_h satisfying*

$$\exists \alpha > 0 \qquad \sup_{v_h \in X_h, v_h \neq 0} \frac{a_h(w_h, v_h)}{\|v_h\|_{X_h}} \geq \alpha \|w_h\|_{X_h} \qquad \forall v_h \in X_h; \tag{38}$$

$$\exists C > 0 \qquad a_h(u_h, v_h) \leq C \|u_h\|_{X_h} \|v_h\|_{X_h} \qquad \forall u_h, v_h \in X_h. \tag{39}$$

Let u and u_h be the solution of

$$\begin{aligned} a(u,v) &= l(v) \quad \forall v \in X, && (40)\\ a_h(u_h,v_h) &= l_h(v) \quad \forall v_h \in X_h. && (41) \end{aligned}$$

Then there exists a constant C not depending on u and h such that

$$\|u-u_h\|_{X_h} \le C\left(\inf_{v_h\in X_h}\|u-v_h\|_{X_h} + \sup_{v_h\in X_h, v_h\neq 0}\frac{(a-a_h)(u,v_h)}{\|v_h\|_{X_h}} + \sup_{v_h\in X_h, v_h\neq 0}\frac{(l-l_h)(v_h)}{\|v_h\|_{X_h}}\right). \qquad (42)$$

Proof. The proof of the above proposition is given in [36]. □

In order to use this result in our problem let us first state the following lemma which shows that the bilinear form $a_h(\cdot,\cdot)$ given in equation (31) satisfies the above mentioned properties.

Lemma 4.5. *Let $a_h(\cdot,\cdot)$ be defined as in (31). Then*

$$\begin{aligned} &\exists \alpha > 0 \qquad \sup_{v_h\in V_h, v_h\neq 0}\frac{a_h(w_h,v_h)}{\|v_h\|_{V_h}} \ge \alpha\|w_h\|_{V_h} \qquad \forall v_h \in V_h; && (43)\\ &\exists C > 0 \qquad a_h(u_h,v_h) \le C\|u_h\|_{V_h}\|v_h\|_{V_h} \qquad \forall u_h, v_h \in V_h. && (44) \end{aligned}$$

Proof. The continuity of $a_h(\cdot,\cdot)$ is a straightforward computation.
To verify the inf-sup condition, we denote by μ the parameter μ_h^H in the first term of the bilinear form $a_h(\cdot,\cdot)$ and we consider the following set

$$X_N(\Omega) = H_0(\mathbf{curl};\Omega)\cap H^0(\mathbf{div};\mu,\Omega) \qquad (45)$$

where $H_0(\mathbf{curl};\Omega)$ denotes the functions f in $H(\mathbf{curl};\Omega)$ satisfying $\mathbf{n}\times f = 0$ on $\partial\Omega$ and $H^0(\mathbf{div};\mu,\Omega) = \{U \in [L^2(\Omega)]^3 : \nabla\cdot(\mu U) = 0\}$. It was shown in [33] that for $v \in X_N(\Omega)$,

$$\|v\|_{H(\mathbf{curl};\Omega)} \le C \sup_{\phi\in X_N(\Omega)}\frac{|a_h(v,\phi)|}{\|\phi\|_{H(\mathbf{curl};\Omega)}} \qquad (46)$$

Let w be in $H_0(\mathbf{curl};\Omega)$ and set $w = v+\nabla\psi$ where $\psi \in H_0^1(\Omega)$ solves

$$(\mu\nabla\psi,\nabla\theta) = (\mu w,\nabla\theta) \quad \text{for all} \quad \theta \in H_0^1(\Omega),$$

so that v is in $X_N(\Omega)$. Thus,

$$\begin{aligned} \|v\|_{H(\mathbf{curl};\Omega)} &\le C\sup_{\phi\in X_N(\Omega)}\frac{|a_h(w,\phi)|}{\|\phi\|_{H(\mathbf{curl};\Omega)}} + C\|\nabla\psi\|_{L^2(\Omega)} && (47)\\ &\le C\sup_{\Theta\in H_0(\mathbf{curl};\Omega)}\frac{a_h(w,\Theta)}{\|\Theta\|_{H(\mathbf{curl};\Omega)}} + C\|\nabla\psi\|_{L^2(\Omega)}. \end{aligned}$$

Now

$$\|\nabla\psi\|^2_{L^2(\Omega)} \le C|a_h(\nabla\psi,\nabla\psi)| = C|a_h(w,\nabla\psi)|,$$

then it follows easily that

$$\|\nabla\psi\|_{H(\mathbf{curl};\Omega)} = \|\nabla\psi\|_{L^2(\Omega)} \le C\sup_{\Theta\in H_0(\mathbf{curl};\Omega)}\frac{|a_h(w,\Theta)|}{\|\Theta\|_{H(\mathbf{curl};\Omega)}}. \qquad (48)$$

The inf-sup condition (43) then follows from the triangle inequality, (47) and (48). □

This lemma plays an important role in evaluating the error of the field $\mathbf{E}^m$. In fact, the weak problem satisfied by $\mathbf{E}^m$, includes some constitutive parameters on which an error has been made when achieving the previous step. Here by using this lemma, we mimic the approach used in case of variational "crime" or approximated quadrature formula, in order to account the approximated value of constitutive parameters. Thus, we have the following result.

Proposition 4.6. *Let $\mathbf{E}^m$ be the solution of (20) and $\mathbf{E}_h^m$ the solution of (30), under the above properties (43-44) on the bilinear forms $a(\cdot,\cdot)$ and $a_h(\cdot,\cdot)$, there exists a constant C not depending on $\mathbf{E}^m$ such that*

$$\|\mathbf{E}^m - \mathbf{E}_h^m\|_V \leq C\left(\inf_{\mathbf{E}_h^{'}\in V_h}\|\mathbf{E}^m - \mathbf{E}_h^{'}\|_V + \max(|\mu^H - \mu_h^H|, |\varepsilon^H - \varepsilon_h^H|)\|\mathbf{E}^m\|_V\right) \tag{49}$$

Proof. By combining the Lemma 4.5 and Proposition 4.4, we have the following inequality

$$\begin{aligned}\|\mathbf{E}^m - \mathbf{E}_h^m\|_V \leq C\Bigg(&\inf_{\mathbf{E}_h^{'}\in V_h}\|\mathbf{E}^m - \mathbf{E}_h^{'}\|_V + \sup_{\mathbf{E}_h^{'}\in V_h, \mathbf{E}_h^{'}\neq 0}\frac{(a-a_h)(\mathbf{E}^m, \mathbf{E}_h^{'})}{\|\mathbf{E}_h^{'}\|_V}\\ &+ \sup_{\mathbf{E}_h^{'}\in V_h, \mathbf{E}_h^{'}\neq 0}\frac{(l-l_h)(\mathbf{E}_h^{'})}{\|\mathbf{E}_h^{'}\|_V}\Bigg)\end{aligned} \tag{50}$$

To obtain the inequality (49), it remains to estimate the second and third terms of (50).

$$(a-a_h)(\mathbf{E}^m, \mathbf{E}_h^{'}) = \int_\Omega ([\mu^H]^{-1} - [\mu_h^H]^{-1})\,\mathbf{curl}\,\mathbf{E}^m \cdot\,\mathbf{curl}\,\mathbf{E}_h^{'} - \omega^2(\varepsilon^H - \varepsilon_h^H)\,\mathbf{E}^m\cdot\,\mathbf{E}_h^{'}\,dx.$$

The use of the Cauchy-Schwarz inequality and the lower boundedness of μ^H (because μ is bounded) allow to obtain

$$|(a-a_h)(\mathbf{E}^m, \mathbf{E}_h^{'})| \leq C\max(|\mu^H - \mu_h^H|, |\varepsilon^H - \varepsilon_h^H|)\|\mathbf{E}^m\|_V\|\mathbf{E}_h^{'}\|_V.$$

We have also

$$(l-l_h)(\mathbf{E}_h^{'}) = \int_{\partial\Omega_N}([\mu^H]^{-1} - [\mu_h^H]^{-1})\,\mathbf{curl}\,\mathbf{E}^m \times \mathbf{n}\cdot\mathbf{E}_h^{'}\,ds.$$

The direct result of Cauchy-Schwarz inequality gives

$$|(l-l_h)(\mathbf{E}_h^{'})| \leq C|[\mu^H]^{-1} - [\mu_h^H]^{-1}|\|\mathbf{curl}\,\mathbf{E}^m \times \mathbf{n}\|_{H^{-1/2}}\|\mathbf{E}_h^{'}\|_{H^{1/2}}.$$

Using the theorem of the trace operator, the last inequality becomes as follows

$$|(l-l_h)(\mathbf{E}_h^{'})| \leq C|[\mu^H]^{-1} - [\mu_h^H]^{-1}|\|\mathbf{E}^m\|_V\|\mathbf{E}_h^{'}\|_V.$$

Then

$$|(l-l_h)(\mathbf{E}_h^{'})| \leq C|\mu^H - \mu_h^H|\|\mathbf{E}^m\|_V\|\mathbf{E}_h^{'}\|_V.$$

according to the lower-boundedness of μ^H. Finally, by combining the above inequalities, the result follows. □

4.4 Step IV: A priori error estimation on the limit field $\mathbf{E}^l$

We recall that the limit field $\mathbf{E}^l$ (resp. $\mathbf{E}_h^l$) is defined as summation of macroscopic $\mathbf{E}^m$ (resp. $\mathbf{E}_h^m$) and corrector field $\mathbf{E}^c$ (resp. $\mathbf{E}_h^c$). Hence,

$$\mathbf{E}^l(x,y) = \mathbf{E}^m(x) + \mathbf{E}^c(x,y) \quad \text{and} \quad \mathbf{E}_h^l(x,y) = \mathbf{E}_h^m(x) + \mathbf{E}_h^c(x,y) \tag{51}$$

It follows that

$$\begin{aligned}\|\mathbf{E}^l - \mathbf{E}_h^l\|_V \leq \|\mathbf{E}^m - \mathbf{E}_h^m\|_V + \sum_{k=1}^{3}\big(&\|\nabla_y(w_k^\varepsilon - w_{k,h}^\varepsilon)\|_{0,Y}\|\mathbf{E}_k^m\|_V\\ &+\|\nabla_y w_{k,h}^\varepsilon\|_{0,Y}\|\mathbf{E}_k^m - \mathbf{E}_{k,h}^m\|_V\big).\end{aligned} \tag{52}$$

Applying the Cauchy-Schwarz inequalities yields

$$\|\mathbf{E}^l - \mathbf{E}_h^l\|_V \leq \|\mathbf{E}^m - \mathbf{E}_h^m\|_V + \left(\|\mathbf{E}^m\|_V^2 + \|w_h^\varepsilon\|_{1,Y}^2\right)^{\frac{1}{2}} \|\mathbf{E}^m - \mathbf{E}_h^m\|_V + \left(\|\mathbf{E}^m\|_V^2 + \|w^\varepsilon\|_{1,Y}^2\right)^{\frac{1}{2}} \|\nabla_y(w^\varepsilon - w_h^\varepsilon)\|_{0,Y} \tag{53}$$

where $w^\varepsilon = (w_1^\varepsilon, w_2^\varepsilon, w_3^\varepsilon)$ and $w_h^\varepsilon = (w_{1,h}^\varepsilon, w_{2,h}^\varepsilon, w_{3,h}^\varepsilon)$. By writing $\|\nabla_y w_{k,h}^\varepsilon\|_{0,Y} \leq \|\nabla_y w_k^\varepsilon\|_{0,Y} + \|\nabla_y(w_{k,h}^\varepsilon - w_k^\varepsilon)\|_{0,Y}$, the using of the error estimate of step I, provided the mesh parameter (h) is bounded, we have the following bound

$$\|\nabla_y w_{k,h}^\varepsilon\|_{0,Y} \leq C\|\nabla_y w_k^\varepsilon\|_{0,Y}.$$

Then equation (53) becomes

$$\|\mathbf{E}^l - \mathbf{E}_h^l\|_V \leq \left(1 + \left(\|\mathbf{E}^m\|_V^2 + \|w^\varepsilon\|_{1,Y}^2\right)^{\frac{1}{2}}\right) \|\mathbf{E}^m - \mathbf{E}_h^m\|_V + \left(\|\mathbf{E}^m\|_V^2 + \|w^\varepsilon\|_{1,Y}^2\right)^{\frac{1}{2}} \|w^\varepsilon - w_h^\varepsilon\|_{1,Y}$$

This shows the following result

Proposition 4.7. *Let $\mathbf{E}^l$ be the limit electric field and $\mathbf{E}_h^l$ the finite element approximated one. Then the error made is given by*

$$\|\boldsymbol{E}^l - \boldsymbol{E}_h^l\|_V \leq \left(1 + \left(\|\mathbf{E}^m\|_V^2 + \|w^\varepsilon\|_{1,Y}^2\right)^{\frac{1}{2}}\right) \|\boldsymbol{E}^m - \boldsymbol{E}_h^m\|_V + \left(\|\mathbf{E}^m\|_V^2 + \|w^\varepsilon\|_{1,Y}^2\right)^{\frac{1}{2}} \|w^\varepsilon - w_h^\varepsilon\|_{1,Y} \tag{54}$$

Let us put all the result of different previous steps devoted to estimate error together. The principal result is presented in the next theorem

Theorem 4.8. *The error made in approximating the limit electric field by conformal finite element method is given by*

$$\begin{aligned}\|\mathbf{E}^l &- \mathbf{E}_h^l\|_V \leq C\left(1 + \left(\|\mathbf{E}^m\|_V^2 + \|w^\varepsilon\|_{1,Y}^2\right)^{\frac{1}{2}}\right) \inf_{\mathbf{E}_h' \in V_h} \|\mathbf{E}^m - \mathbf{E}_h'\|_V \\ &+ C\left(1 + \left(\|\mathbf{E}^m\|_V^2 + \|w^\varepsilon\|_{1,Y}^2\right)^{\frac{1}{2}}\right) \|\mathbf{E}^m\|_V \left(\inf_{w \in U_h} \|w^\mu - w\|_{1,Y} + \inf_{w \in U_h} \|w^\varepsilon - w\|_{1,Y}\right) \\ &+ C\left(\|\mathbf{E}^m\|_V^2 + \|w^\varepsilon\|_{1,Y}^2\right)^{\frac{1}{2}} \inf_{w \in U_h} \|w^\mu - w\|_{1,Y}\end{aligned} \tag{55}$$

where C is a constant not depending neither on $\mathbf{E}^m$ nor w^μ and w^ε.

Remark

It follows from Theorem 4.8 that the error made, when using conformal finite element approximation, is optimal with respect to the approximability of the discrete spaces. This result was obtained in a more general setting, assuming only conformal approximation. Let us now recall some approximability properties of spaces U_h and V_h. This result will give the error made in case of uniform mesh (i.e. relates the error on the mesh parameter). This will end the proof of the a priori error estimate. The approximation property of V_h is given by Lemma 4.9 ([34]).

Lemma 4.9. *Let τ_h be a regular mesh on Ω. Then if $\mathbf{u} \in (H^s(\Omega))^3$ and $\mathbf{curl\ u} \in (H^s(\Omega))^3$ for some $1/2 + \delta \leq s \leq n$ for $\delta > 0$ then*

$$\|\mathbf{u} - r_h\mathbf{u}\|_{(L^2(\Omega))^3} + \|\mathbf{curl}\ (\mathbf{u} - r_h\mathbf{u})\|_{(L^2(\Omega))^3} \leq Ch^s\left(\|\mathbf{u}\|_{(H^s(\Omega))^3} + \|\mathbf{curl\ u}\|_{(H^s(\Omega))^3}\right) \tag{56}$$

where $r_h : (H^s(\Omega))^3 \to V_h$ is the global interplant operator.

Let us announce the following standard result [35].

Lemma 4.10. *Given a conforming shape-regular mesh τ_h, for $u \in H^s(\Omega)$, and*

$$\frac{3}{2} < s \leq m+1, \quad 0 \leq m \leq s,$$

there exists a constant C, depending only on m and s such that

$$\inf_{u_h \in V_h} |u - u_h|_{H^m(\Omega)} \leq Ch^{s-m} |u|_{H^s(\Omega)}. \tag{57}$$

Now we can put together all the above result to end the a priori estimation.

Theorem 4.11. *Let $\mathbf{E}^m \in (H^s(\Omega))^3$ be the exact macroscopic electric field and $\mathbf{E}^c$ the corrector field. Assuming that the term w^β (25) is approximated by nodal Lagrange conformal finite element on a shape-regular mesh of characteristic size h_Y, and that the macroscopic electric field is approximated by the first order Nédélec finite element method on a shape regular mesh of characteristic size h_Ω. Then, there exists a constant C depending only on $\mathbf{E}^m, w$, and s such that*

$$\|\mathbf{E}^l - \mathbf{E}^l_h\|_{(L^2(\Omega))^3} + \|\mathbf{curl}(\mathbf{E}^l - \mathbf{E}^l_h)\|_{(L^2(\Omega))^3} \leq C[h^\gamma_\Omega + h_Y], \tag{58}$$

with $\gamma = \min(2, s)$. Furthermore, if the mesh size of the whole domain is a constant factor of the mesh size of the periodic cell (i.e., $h_\Omega = \kappa h_Y$ for some constant κ), then

$$\|\mathbf{E}^l - \mathbf{E}^l_h\|_{(L^2(\Omega))^3} + \|\mathbf{curl}(\mathbf{E}^l - \mathbf{E}^l_h)\|_{(L^2(\Omega))^3} \leq Ch^\gamma(\Omega), \tag{59}$$

with $\gamma = \min(1, s)$.

Proof. It is a combination of the approximation property in the space V_h using lemma 4.9 and Lemma 4.10 together with Theorem 4.8. □

5. Numerical results

In this section, we present the numerical results of the error estimate related to HCPs (ε^H, μ^H) and the macroscopic electric field $\mathbf{E}^m$. The studied material is periodic and the purpose is to examine the influence of mesh size, h, on the numerical results.

5.1 HCPs ε^H and μ^H

We consider a unit cell $Y = (0,1)^3$ occupied by the laminated materials which composed by two components. These components have the thickness l_1 and l_2 and characterized by two different permittivities $(\varepsilon_1, \varepsilon_2)$ and permeabilities (μ_1, μ_2) (Fig.1).

$$\varepsilon_1 = 10\varepsilon_0, \quad \varepsilon_2 = 5\varepsilon_0, \quad \mu_1 = 1\mu_0, \quad \mu_2 = 10\mu_0 \quad \text{and} \quad l_1 = l_2 = 0.5cm$$

where ε_0 and μ_0 are, respectively, the permittivity and the permeability of the vacuum.
Our choice of the laminated material in this study is dictated by the possibility to express the exact expression of the HCPs. We denote that in the other cases, to find the exact expression of ε^H and μ^H is not possible, for example, the material composed by the inclusions with complicated shape suspended in the host media.

If the response of a composite material changes with the direction of the excitation by electromagnetic wave, this material is anisotropic. In spite of the permittivity and the permeability of the anisotropic material are constant, the expression of its HCPs are not constant but diagonal matrices of the form

$$\varepsilon^H = \begin{pmatrix} \varepsilon^H_x & 0 & 0 \\ 0 & \varepsilon^H_y & 0 \\ 0 & 0 & \varepsilon^H_z \end{pmatrix} \qquad \mu^H = \begin{pmatrix} \mu^H_x & 0 & 0 \\ 0 & \mu^H_y & 0 \\ 0 & 0 & \mu^H_z \end{pmatrix}$$

In the case of the laminated material (Fig.1), the expression of HCPs' z-components ε^H_z and μ^H_z of ε^H and μ^H, respectively, is given analytically by the thickness of the component of material l_1, l_2 and its parameters

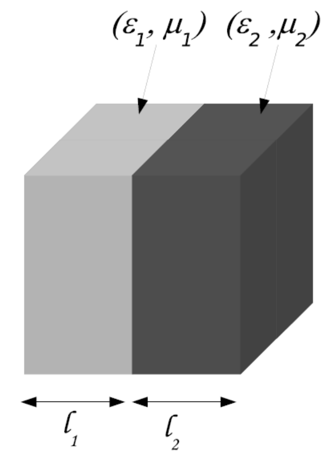

Figure 1. *Unit cell of laminate material containing two components.*

$(\varepsilon_1, \varepsilon_2)$ and (μ_1, μ_2).

$$\frac{l_1 + l_2}{\varepsilon_z^H} = \frac{l_1}{\varepsilon_1} + \frac{l_2}{\varepsilon_2} \tag{60}$$

$$\frac{l_1 + l_2}{\mu_z^H} = \frac{l_1}{\mu_1} + \frac{l_2}{\mu_2} \tag{61}$$

We now investigate the precision of the proposed error estimation by considering the dependence of the relative error $(\varepsilon_z^H - \varepsilon_{z,h}^H)/\varepsilon_z^H$ and $(\mu_z^H - \mu_{z,h}^H)/\mu_z^H$, respectively, for the HCPs $(\varepsilon_z^H, \mu_z^H)$ on the mesh parameter h. The approximated parameters ε_h^H and μ_h^H are expressed respectively as function of w_h^ε and w_h^μ solutions of problem (25). The terms w_h^ε and w_h^μ are computed in the unit cell and the mesh is decomposed by regular tetrahedra elements. The computation is carried out at four values of $h = h_i$ $(i = 1, ..., 4)$ with $h_i = 1/2^i cm$. Using the bi-conjugate gradient and incomplete LU (ILU) preconditioning in each computation the solution is obtained. In Fig. 2 and Fig. 3, we present the relative error of $\varepsilon_{z,h}^H$ and $\mu_{z,h}^H$ respectively as function of the mesh parameter h. The obtained numerical results decrease linearly with the parameter h, as predicted by the theoretical analysis.

5.2 Electric field

In this subsection, we are going to analyze the macroscopic field by making the comparison between the "exact solution" $(\mathbf{E}^m)$ and the numerical one $(\mathbf{E}_h^m)$ computed in different meshes. For this, we consider a bounded domain Ω which characterized by (ε^H, μ^H) (Fig. 4). The size of the Ω is $a \times a \times L$ with $a = 2cm$ and $L = 3cm$. The incident field, $\mathbf{E}_i^m$, is applied to surface $S_1 = $ (abcd) and is assumed to satisfy the Maxwell system. On the surface $S_2 = $ (efgh) of the boundary $\partial\Omega$, the opposite surface of S_1, we impose the perfect conducting boundary condition. In addition, We need distinguish a given incident field and resulting reflected field by the surface S_2. The reflected field is denoted by $\mathbf{E}_r^m$. A typical example might be the plane wave given by

$$\mathbf{E}_i^m = \mathbf{p}_i \exp(-ik\, \mathbf{x} \cdot \mathbf{d}) \tag{62}$$

$$\mathbf{E}_r^m = \mathbf{p}_r \exp(ik\, \mathbf{x} \cdot \mathbf{d}) \tag{63}$$

where k is the wavenumber, $\mathbf{d} \in \mathbb{R}^3$ is a unit vector giving the direction of propagation of the wave, and the vector $\mathbf{p}_j \neq 0$ with $(j \in \{i, r\})$ is called the polarization vector . This vector must be orthogonal to the direction of propagation, so $(\mathbf{p}_j \cdot \mathbf{d} = 0)$, $(j \in \{i, r\})$. The total field $\mathbf{E}_T^m$ consists of the incident field $\mathbf{E}_i^m$ and the reflected field $\mathbf{E}_r^m$

$$\mathbf{E}_T^m = \mathbf{E}_i^m + \mathbf{E}_r^m$$

The incident wave is applied at the frequency $f = 1GHz$ and its vector propagation is polarized along the z-axis $(\mathbf{d} = (0, 0, 1))$. The vector of polarization $\mathbf{p}_j$ is along the y-axis and the magnetic field $(\mathbf{H}_T^m = \alpha\ \mathbf{curl}\ \mathbf{E}_T^m)$ is according the x-axis. We impose that on S_1, the total field $\mathbf{E}_T^m$ equal to $(0, 1, 0)$. The field $\mathbf{E}_T^m$ is perpendicular

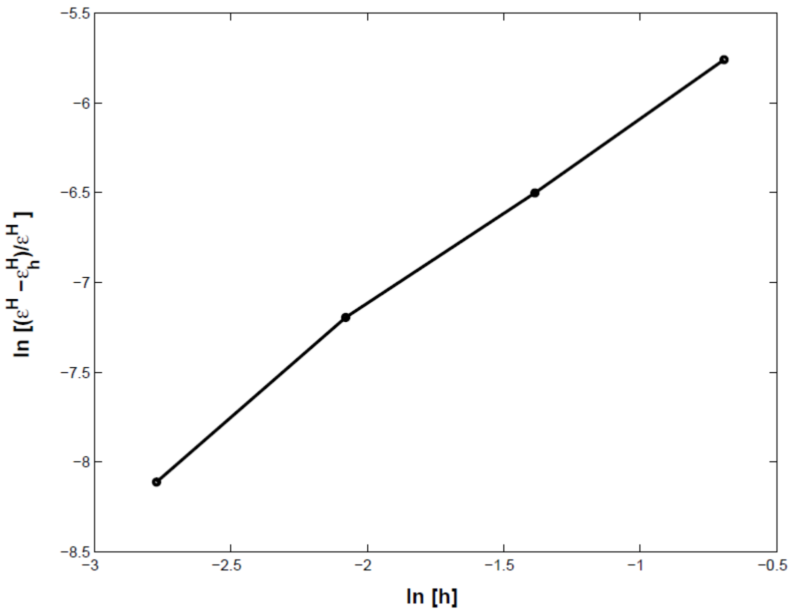

Figure 2. *Relative error (in logarithm scale) on* ε^H

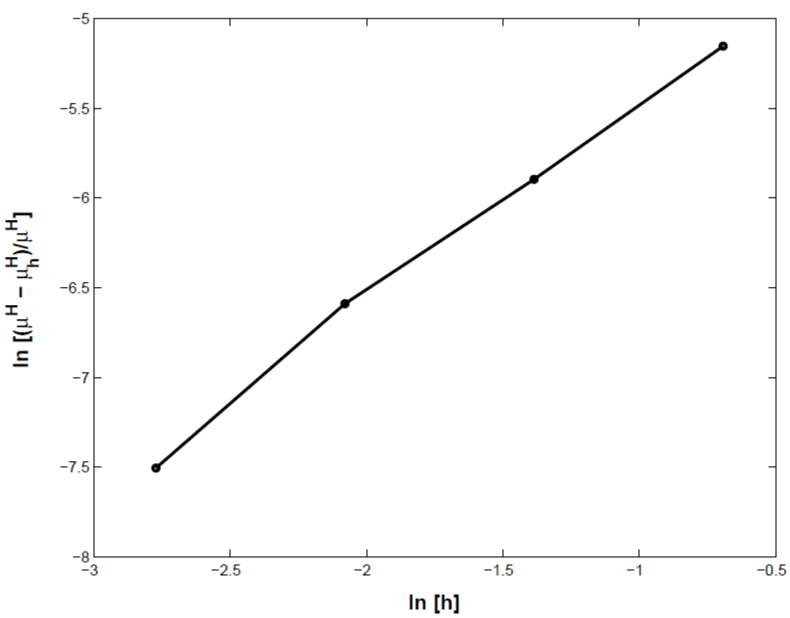

Figure 3. *Relative error (in logarithm scale) on* μ^H

to the surface $S_3 = (\text{bcgf})$ and $S_4 = (\text{adhe})$ (see Fig. 4). It is also tangential to the surfaces $S_5 = (\text{abfe})$ and $S_6 = (\text{dcgh})$. So, it verifies the following problem

$$\left\{ \begin{array}{lcl} \mathbf{curl}([\mu^H]^{-1}\mathbf{curl}\ \mathbf{E}_T^m) - \varepsilon^H \omega^2 \mathbf{E}_T^m & = & 0, \text{ in } \Omega \\ \mathbf{n} \times \mathbf{E}_T^m & = & \mathbf{n} \times \mathbf{g} \text{ on } S_1 \\ \mathbf{n} \times \mathbf{E}_T^m & = & 0 \text{ on } S_2, S_3, S_4 \\ \mathbf{n} \times \mathbf{curl}\ \mathbf{E}_T^m & = & 0 \text{ on } S_5, S_6 \end{array} \right. \tag{64}$$

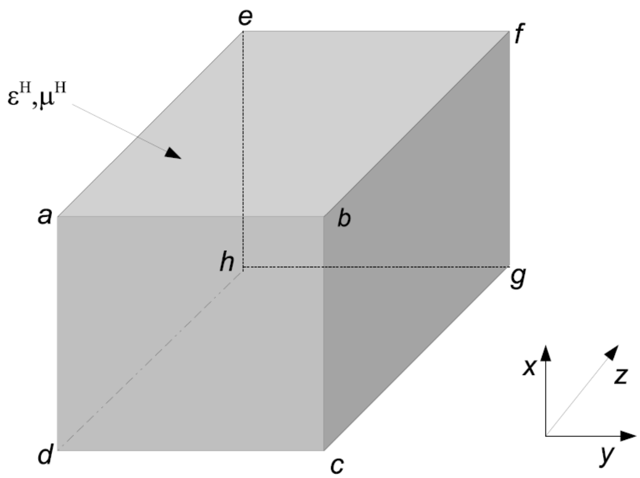

Figure 4. *Bounded homogenized domain characterized by HCPs* (ε^H, μ^H)

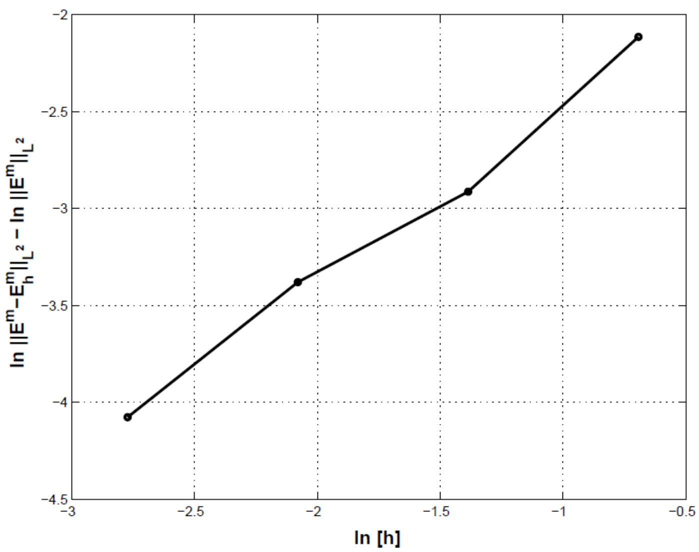

Figure 5. *Relative error of* $\boldsymbol{E}_h^m$

Here, **g** results from a given incidence field. We can easily find that the exact solution of this problem can be expressed as follows

$$
\begin{aligned}
\mathbf{E}^m_{T,x}(z) &= 0, \\
\mathbf{E}^m_{T,y}(z) &= \sin(k(L-z))/\sin(kL), \\
\mathbf{E}^m_{T,z}(z) &= 0.
\end{aligned}
$$

The numerical solution $\mathbf{E}^m_{T,h}$ of the problem (64) is obtained by using the Nédélec finite elements method and the triangulation are the tetrahedra. The field $\mathbf{E}^m_{T,h}$ is computed at four values of the $h = 1/2^i cm$ $(i = 1, ..., 4)$. In this case, the size mesh of Ω is identical to the size of Y $(h_\Omega = h_Y)$ for each value of h. The inversion of the obtained linear system is carried out by the bi-conjugate gradient solver combined with the incomplete LU (ILU) preconditioning on each mesh.

In Fig. 5, we present the variation of the relative error $\|\mathbf{E}^m - \mathbf{E}^m_h\|_{L^2(\Omega)}/\|\mathbf{E}^m\|_{L^2(\Omega)}$ of the macroscopic electric field $\mathbf{E}_h^m$, as function of the parameter h. The obtained numerical results confirm the theoretical linear dependence of macroscopic field on the parameter h.

6. Conclusion

We have presented the a priori error estimation of the electromagnetic properties obtained using two-scale convergence or unfolding method. These properties are the homogenized constitutive parameters (HCPs), the macroscopic field and the limit electromagnetic field in 3D periodic structure. The HCPs are approximated respectively by using the Lagrange and the first order Nédélec conforming finite element method. We note that the approximation of limit field is obtained from those of HCPs and macroscopic field. The optimality of the convergence is obtained for these electromagnetic quantities and the numerical results are also presented which confirm the theoretical results.

References

[1] ALLAIRE, Grégoire. Homogenization and two-scale convergence. *SIAM Journal on Mathematical Analysis*, 1992, vol. 23, no 6, p. 1482-1518.

[2] AMIRAT, Youcef, HAMDACHE, Kamel, et ZIANI, Abdelhamid. Homogenization of degenerate wave equations with periodic coefficients. *SIAM Journal on Mathematical Analysis*, 1993, vol. 24, no 5, p. 1226-1253.

[3] BENSOUSSAN, Alain, LIONS, Jacques-Louis, et PAPANICOLAOU, George. *Asymptotic Analysis for Periodic Structures* North-Holland,Amsterdam, 1978.

[4] CONCA, Carlos et VANNINATHAN, Muthusamy. Homogenization of periodic structures via Bloch decomposition. *SIAM Journal on Applied Mathematics*, 1997, vol. 57, no 6, p. 1639-1659.

[5] DUVAUT, Georges et LIONS, Jacques Louis. *Inequalities in mechanics and physics*. Springer, Berlin, 1976.

[6] NGUETSENG, Gabriel. A general convergence result for a functional related to the theory of homogenization. *SIAM Journal on Mathematical Analysis*, 1989, vol. 20, no 3, p. 608-623.

[7] OUCHETTO, Ouail, OUCHETTO, Hassania, ZOUHDI, Said, et al. Homogenization of Maxwell's Equations in Lossy Biperiodic Metamaterials. *Antennas and Propagation, IEEE Transactions on*, 2013, vol. 61, no 8, p. 4214-4219.

[8] OUCHETTO, Ouail et ESSAKHI, Brahim. Frequency Domain Homogenization of Maxwell Equations in Complex Media. *International Journal of Engineering and Mathematical Modelling*, 2015, vol. 2, no 1, p. 1-15.

[9] OUCHETTO, Ouail. *Modélisation large bande de métamatériaux bianisotropes et de surfaces structurées*. 2006. Thèse de doctorat. Paris 11.

[10] SÁNCHEZ-PALENCIA, Enrique. Non-homogeneous media and vibration theory. In : *Non-homogeneous media and vibration theory*. 1980.

[11] SHKOLLER, S. et HEGEMIER, G. Homogenization of plane wave composite using two-scale convergence, *Int. J. Solids Structures*, 1995, vol 32, no 6/7, p. 783-794.

[12] BRUN, Michele, GUENNEAU, Sébastien, et MOVCHAN, Alexander B. Achieving control of in-plane elastic waves. *Applied Physics Letters*, 2009, vol. 94, no 6, p. 061903.

[13] RAINSFORD, Tamath J., MICKAN, Samuel P., et ABBOTT, Derek. T-ray sensing applications: review of global developments. In : *Smart Materials, Nano-, and Micro-Smart Systems. International Society for Optics and Photonics*, 2005. p. 826-838.

[14] COTTON, Michael G. *Applied Electromagnetics*. 2003 Technical Progress Report (NITA – ITS) (Boulder, CO, USA: NITA – Institute for Telecommunication Sciences) Telecommunications Theory (3): 4–5, 2003.

[15] ALICI, Kamil Boratay et OZBAY, Ekmel. Radiation properties of a split ring resonator and monopole composite. *Physica Status Solidi-B-Basic Solid State Physics*, 2007, vol. 244, no 4, p. 1192.

[16] PENDRY, John Brian. Negative refraction makes a perfect lens. *Physical review letters*, 2000, vol. 85, no 18, p. 3966.

[17] WUTTIG, Matthias et YAMADA, Noboru. Phase-change materials for rewriteable data storage. *Nature materials*, 2007, vol. 6, no 11, p. 824-832.

[18] SINGH, Ranjan, ROCKSTUHL, Carsten, LEDERER, Falk, et al. Coupling between a dark and a bright eigenmode in a terahertz metamaterial. *Physical Review B*, 2009, vol. 79, no 8, p. 085111.

[19] BARBATIS, G. et STRATIS, I. G. Homogenization of Maxwell's equations in dissipative bianisotropic media. *Mathematical methods in the applied sciences*, 2003, vol. 26, no 14, p. 1241-1253.

[20] KRISTENSSON, Gerhard et WELLANDER, Niklas. Homogenization of the Maxwell equations at fixed frequency. *SIAM Journal on Applied Mathematics*, 2003, vol. 64, no 1, p. 170-195.

[21] BOSSAVIT, Alain, GRISO, Georges, et MIARA, Bernadette. Modelling of periodic electromagnetic structures bianisotropic materials with memory effects. *Journal de mathématiques pures et appliquées*, 2005, vol. 84, no 7, p. 819-850.

[22] POUPAUD, F. et MARKOWICH, P. A. The Maxwell equation in a periodic medium; homogenization of the energy density. *Ann. Scuola Norm. Sup. Pisa cl. Sci-(4)*, vol. 23, p. 301-324.

[23] WELLANDER, Niklas. Homogenization of the Maxwell equations: Case I. Linear theory. *Applications of Mathematics*, 2001, vol. 46, no 1, p. 29-51.

[24] WELLANDER, Niklas. Homogenization of the Maxwell equations: Case II. Nonlinear conductivity. *Applications of Mathematics*, 2002, vol. 47, no 3, p. 255-283.

[25] OUCHETTO, Ouail, ZOUHDI, Saïd, BOSSAVIT, Alain, et al. Modeling of 3-D periodic multiphase composites by homogenization. *Microwave Theory and Techniques, IEEE Transactions on*, 2006, vol. 54, no 6, p. 2615-2619.

[26] OUCHETTO, O., ZOUHDI, S., BOSSAVIT, A., et al. Homogenization of structured electromagnetic materials and metamaterials. *Journal of materials processing technology*, 2007, vol. 181, no 1, p. 225-229.

[27] OUCHETTO, Ouail, QIU, Cheng-Wei, ZOUHDI, Saïd, et al. Homogenization of 3-d periodic bianisotropic metamaterials. *Microwave Theory and Techniques, IEEE Transactions on*, 2006, vol. 54, no 11, p. 3893-3898.

[28] MIARA, B., ROHAN, E., GRISO, G., et al. Application of multi-scale modelling to some elastic, piezoelectric and electromagnetic composites. *Mechanics of advanced materials and structures*, 2007, vol. 14, no 1, p. 33-42.

[29] OUCHETTO, O., ZOUHDI, S., RAZEK, A., et al. Effective constitutive parameters of structured chiral metamaterials. *Microwave and optical technology letters*, 2006, vol. 48, no 9, p. 1884-1886.

[30] OUCHETTO, O., ZOUHDI, S., BOSSAVIT, A., et al. Homogenization of 3d structured composites of complex shaped inclusions. In : *Progress Electromagn. Res. Symp.* 2005. p. 112.

[31] OUCHETTO, O., ZOUHDI, S., BOSSAVIT, A., et al. Effective constitutive parameters of periodic composites. In : *Microwave Conference, 2005 European. IEEE*, 2005. p. 2 pp.

[32] OUCHETTO, O., ZOUHDI, S., BOSSAVIT, A., et al. A new approach for the homogenization of three-dimensional metallodielectric lattices: the periodic unfolding method. In: *PECS-VI, International Symposium on Photonic and Electromagnetic Crystal Structures*, June 2005.

[33] BRAMBLE, James et PASCIAK, Joseph. Analysis of a finite PML approximation for the three dimensional time-harmonic Maxwell and acoustic scattering problems. *Mathematics of Computation*, 2007, vol. 76, no 258, p. 597-614.

[34] MONK, Peter. *Finite element methods for Maxwell's equations*. Oxford University Press, 2003.

[35] CIARLET, Philippe G. *The finite element method for elliptic problems*. Siam, 2002.

[36] ERN, Alexandre et GUERMOND, Jean-Luc. *Theory and practice of finite elements*. Springer Science & Business Media, 2013.